M.Sc. PHYSICS –I YEAR

DKP14 : ELECTROMAGNETIC THEORY

SYLLABUS

UNIT I Electrostatics

Coulomb's Law– Charge distributions– Lines of force and flux–Gauss's Law and its applications –The potential function– Poission's equation and Laplace equation– Equipotential surfaces– Field due to continues charge distribution– Energy associated to an electrostatic field– Electrostatic uniqueness theorem.

UNIT II Magnetostatics

Lorentz force – Faraday's law – Magnetic field strength and Ampere's circuital law– Biot-Savart's law – Ampere's force law – Magnetic vector potential – Equation of continuity–The far magnetic field of a current distribution– Magnetic field due to volume distribution of current

UNIT III Dielectrics;

Polarization – the electric field inside a dielectric medium – Gauss law in dielectric and the electric displacement – Electric susceptibility and dielectric constant – Boundary conditions on the field vectors – Dielectric sphere in a uniform electric field– Force on a point charge embedded in a dielectric

UNIT IV Maxwell's equation and propagation of EM waves:

Maxwell's equations and their physical significance – Plane wave equation in homogeneous medium and in free space – relation between E and H vectors in a uniform plane wave– The wave equation for a conducting medium – Skin depth – Wave propagation in dielectric– Poynting vector – Poynting's theorem

UNIT V Waves in bounded region and Radiation

Reflection and refraction of EM waves at the boundary of two conducting media – Normal incidence and oblique incidence – Brewster's angle– Wave guides – Rectangular wave guide – Cavity resonators – Radiation from and oscillating dipole –Transmission line theory – Transmission line as distribution circuit– Basic transmission line equations

Books for Study and Reference

1. Foundation of EMT – Third edition –John R. Reity, Frederick J. Milford and Robert W. Christy.
2. Electromagnetic theory – Prabir K. Basu and HrishikeshDhasmana.
3. Introduction to Electrodynamics– David J Griffiths.
4. Electromagnetic fields and waves– P.Lorrain and D.Corson.
5. Electrodynamics– B.P.Laud.

UNIT I : ELECTROSTATICS

Coulomb's Law – Charge distributions – Lines of force and flux – Gauss law and its applications – The potential function – Poisson's equation and Laplace equation – Equipotential surfaces – Field due to continuous charge distribution – Energy associated to an electrostatic field – Electrostatic uniqueness theorem.

1.1 Introduction

The theory which describes physical phenomena related to the interaction between stationary electric charges or charge distributions in space with stationary boundariesis called *electrostatics*. The forces on all stationary charges are purely electrical. Charge is the fundamental characteristic property of the elementary particle that constitutes matter. The fundamental law for electrostatic forces is Coulomb's law. The basic concepts and other laws of electrostatic can be derived from it.

1.2Coulomb's Law

According to Coulomb, the force of attraction or repulsion between two electric point charges is directly proportional to the product of magnitude of two charges and inversely proportional to the square of the distance between them. The direction of force is always along the line joining the two charges. For the charges of opposite sign, the force is attractive while that for the same sign, it is repulsive.

Consider two point charges q and Q separated by a distance r (Figure 1.1a). Mathematically the force F of interaction between two point charges is given by

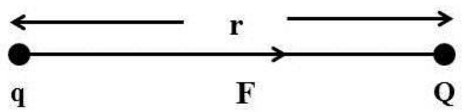


Figure 1.1a

$$F \alpha \frac{qQ}{r^2}$$

$$F = K\,\frac{qQ}{r^2}$$

where K is known as proportionality constant and is equal to $1/4\pi\varepsilon_0$, hence we write

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r} \qquad \text{--- 1.1}$$

The equation 1.1 is known as Coulomb's law of force in electrostatics.

Statement: The force of attraction or repulsion between any two point charges is directly proportional to the product of magnitude of charges and inversely proportional to the square of the distance between them.

The constant $\varepsilon_0$is called the permittivity of free space. In MKS units, the force, distance and charge are measured in Newton, metre and Coulomb respectively. Thus ε_0= $8.85 \times 10^{-12}\ C^2\text{-}N^{-1}\text{-}m^{-2}$.

1.3 The Electric field strength

The region surrounding any electric charge or group of point charges, in which the effect of its electrostatic force can be experienced is called an electric field. The strength of electric field i.e. electric field strength (E) is defined as the force experienced by a unit positive test charge placed at that.

Consider some point charges q_1, q_2, $q_{3,}$ …….q_n at distances r_1, r_2,…….r_n from the test charge Q as shown in Figure 1.1b.

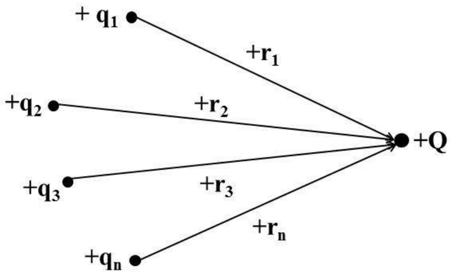


Figure 1.1b

According to the principle of superposition, the total force acting on a test charge due to all other charges is

$$F = F_1 + F_1 + F + \cdots\cdots \qquad \text{--- 1.2}$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 Q}{r_1^2} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 Q}{r_2^2} + \cdots\cdots$$

$$F = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 Q}{r_1^2} + \frac{q_2 Q}{r_2^2} + \cdots\cdots\right]$$

$$F = \frac{Q}{4\pi\varepsilon_0}\left[\frac{q_1}{r_1^2} + \frac{q_2}{r_2^2} + \cdots\cdots\right]$$

$$F = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2} \qquad \text{---1.3}$$

$$F = QE \qquad \text{--- 1.4}$$

$$\text{where } E = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^{\,2}} \qquad \text{--- 1.5}$$

or
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^{\,2}}\hat{r}_i$$

is called the electric field strength. The electric field varies from point to point. The electric field intensity can be defined as the force per unit charge and is measured in Newton/Coulomb.

1.4 Charge distributions

The forces and electric fields due to point charges are only considered so far. But practically in addition to point charges, there is a possibility for the existence of continuous charge distributions along a line, on a surface or in a volume. Thus there are four types of charge distributions, which are

a. Point charge

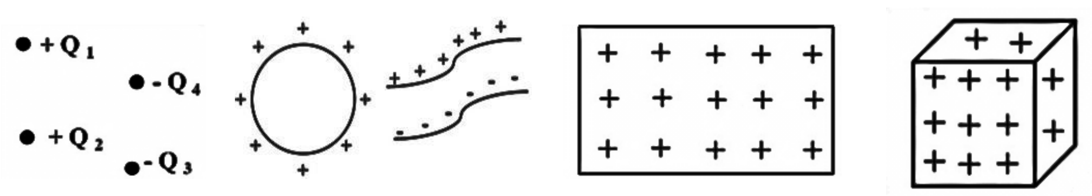
b. Line charge

c.Surface charge and

d. Volume charge

a. Point charge distribution

If the dimension of a surface carrying charge is very small compared to the region surrounding it, then the surface can be treated as a point and the corresponding charge is called a point charge. The point charge has a definite position but not a dimension as shown in Figure 1.2a. The point charge can be either positive or negative.



(a) (b) (c) (d)

Figure 1.2

b. Line charge distribution

It is possible that the charge may be extended along a line. Such a uniform distribution of charge along a line is called a line charge. This is shown in Figure 1.2b.The line charge density is denoted by λ and is defined as charge per unit length.

$$\lambda = \frac{\text{Charge}}{\text{Unit length}} \quad \text{Coulomb / metre}$$

c. Surface charge distribution

If the charge is distributed uniformly over a two dimensional surface then it is called a sheet of charge or a surface charge. The surface charge is shown in Figure 1.2c. The surface charge density is denoted by σ and is defined as charge per unit surface area.

$$\sigma = \frac{\text{Charge}}{\text{Unit area}} \quad \text{Coulomb / metre}^2$$

d. Volume charge distribution

If the charge is distributed uniformly in a volume then it is called volume of charge. The volume charge is shown in Figure 1.2d. The volume charge density is denoted by ρ and is defined as charge per unit volume.

$$\rho = \frac{\text{Charge}}{\text{Unit volume}} \quad \text{Coulomb / metre}^3$$

1.5 Field due to continuous charge distributions

The definition of the electric field i.e. equation 1.5 assumes that the source charge is a set of discrete point charges q_i. However, we may consider these charges as the continuous charge distribution over some region (i.e. distributed continuously over some region).

In each case, we can divide the total charge into several infinitesimal parts, each of which can be considered as a point charge. We thus represent the total charge as a continuous collection of point charges and obtain the field intensity at any point due to the total charge as the vector superposition of the field intensities due to individual point charges. However, now we have to evaluate integrals instead of summation of a few terms since the distribution of charges is continuous as a substitute of being discrete. Hence in equation 1.5, the summation may be replaced by an integral i.e.

$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r} \qquad \text{--- 1.6}$$

1.5.1 Electric field due to line charge

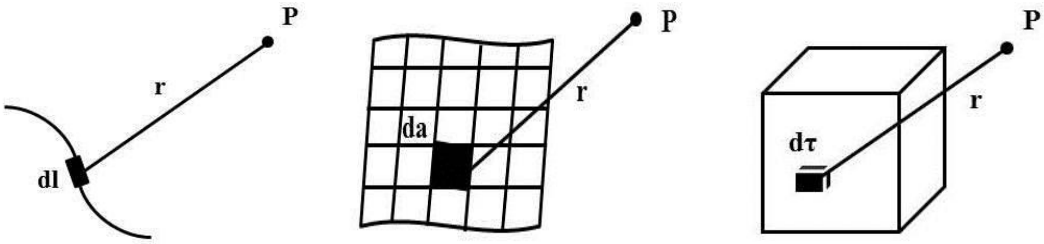
Consider a line charge distribution having a linear charge density λ as shown in Figure 1.3a. The charge dq on the differential length dl is λdl.

Hence the electric field dE at a point P due to dq is given by

$$dE = \frac{dq}{4\pi\varepsilon_0 r^2}\ \hat{r} = \frac{\lambda dl}{4\pi\varepsilon_0 r^2}\ \hat{r} \qquad \text{--- 1.7}$$

The total electric field at a point P due to line charge can be obtained by integrating eqn. 1.7 over the length of the charge

$$E = \frac{1}{4\pi\varepsilon_0} \int_{Line} \frac{\hat{r}}{r^2} [\lambda dl] \qquad \text{---1.8}$$



(a) (b) (c)

Figure 1.3

1.5.2 Electric field due to surface charge

Consider a surface charge distribution having a surface charge density σ as shown in Figure 1.3b. The charge dq on the differential surface area ds is σds.

Hence the electric field dE at a point P due to dq is given by

$$dE = \frac{dq}{4\pi\varepsilon_0 r^2}\ \hat{r}\ = \frac{\sigma ds}{4\pi\varepsilon_0 r^2}\ \hat{r} \qquad \text{--- 1.9}$$

The total electric field E at a point P due to surface charge can be obtained by integrating equation 1.9 over the surface area is

$$E = \frac{1}{4\pi\varepsilon_0} \int_{Surface} \frac{\hat{r}}{r^2} [\sigma ds] \qquad \text{--- 1.10}$$

1.5.3 Electric field due to volume charge

Consider a volume charge distribution having a volume charge density ρ as shown in Figure 1.3c. The charge dq on the differential volume dτ is ρdτ.Hence the electric field dE at a point P due to dq is given by

$$dE \frac{dq}{4\pi\varepsilon_0 r^2}\ \hat{r} = \frac{\rho d\tau}{4\pi\varepsilon_0 r^2}\ \hat{r} \qquad \text{--- 1.11}$$

The total electric field at a point P due to surface charge can be obtained by integrating eqn. 1.11 over the volume in which charge is accumulated.

$$E = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\hat{r}}{r^2} [\rho d\tau] \qquad \text{---1.12}$$

Thus all type of charge distribution is possible. Therefore the total electric field at a point is the vector sum of individual electric field intensities produced by all charges at a point. Hence

$$E_{Total} = E_P + E_L + E_S + E_V \qquad \text{--- 1.13}$$

where E_P, E_L, E_S and E_V are the field intensities due to point, line, surface and volume charge distributions respectively.

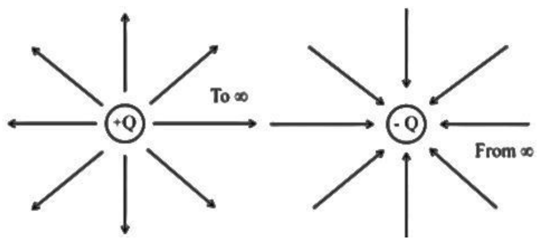
1.6Lines of force and flux

The concept of lines of force is a convenient way to express the electric fieldE. A line of fore is an imaginary line or curve drawn in such a way that its direction at any point gives the direction of the electric field at that point. The total number of lines per unit cross sectional area is called as the flux density and is proportional to the magnitude of the electric field E. The number of flux lines also depends on the magnitude of the charge. The flux density is stronger in the region where the field lines are very close and is weaker in the region where the field lines are far apart.

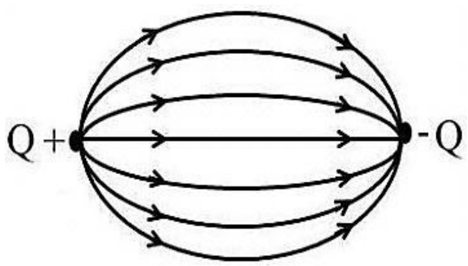
The properties of the filed lines are summarized as follows.

1. They emanate from a positive point charge an end at a negative point charge symmetrically in all possible directions (Figure 1.4a).

2. They originate on positive charges and terminate on negative charges (Figure 1.4b).



(a)



(b)

Figure 1.4

3. They can never cross each other.

4. There is more number of lines if electric field is stronger.

5. The lines are independent of the medium in which charges are placed.

6. The lines always enter or leave the charged surface normally.

1.7 Gauss law and its applications

Gauss law is the inverse of Coulomb's law. By Coulomb's law, one can calculate the electric field E for a given charge. But the Gauss's law enables us to determine charge only if E is known.

Gauss's law gives a relation between the flux over any closed surface, called Gaussian surface and the total charge enclosed within the surface.

Statement: "The total electric flux over any closed surface is equal to $(1/\varepsilon_0)$ times the total charge enclosed within the surface".

Proof: As the electric field strength is proportional to the number of lines per unit cross sectional area (i.e. area perpendicular to the direction of lines of force), the flux of E(i.e. ∫ E·da) through any surface is proportional to the number of field lines passing through that surface.

Now let us consider a point charge placed in the origin, and then the flux of E through a sphere of radius r is

$$\int_{Surface} E\cdot ds = \int_{Surface} \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot ds \qquad \text{---1.14}$$

According to spherical coordinates, $ds = \left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right)$, hence

$$\int_{Surface} E\cdot ds = \int_{Surface} \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right) \qquad \text{--- 1.15}$$

$$\int_{Surface} E\cdot ds = \int_{Surface} \frac{1}{4\pi\varepsilon_0}(q\hat{r})\cdot\left(\sin\theta \, d\theta \, d\phi \, \hat{r}\right)$$

$$\int_{Surface} E\cdot ds = \int_{Surface} \frac{q}{4\pi\varepsilon_0}\cdot\left(\sin\theta \, d\theta \, d\phi \, \hat{r}\hat{r}\right)$$

$$\int_{Surface} E\cdot ds = \int_{Surface} \frac{q}{4\pi\varepsilon_0}\cdot\left(\sin\theta \, d\theta \, d\phi\right)$$

$$\int_{Surface} E\cdot ds = \frac{q}{4\pi\varepsilon_0}\cdot\int_{Surface}\left(\sin\theta \, d\theta \, d\phi\right)$$

$$\int_{Surface} E \cdot ds = \frac{q}{4\pi\varepsilon_0} 4\pi \quad \left[\because \int_{Surface} (\sin\theta \, d\theta \, d\phi) = 4\pi \right]$$

$$\int_{Surface} E \cdot ds = \frac{q}{\varepsilon_0}$$

$$\int_{Surface} E \cdot ds = \left(\frac{1}{\varepsilon_0}\right) q \qquad \text{--- 1.16}$$

From equation 1.16 we note that the radius of the sphere cancels out, i.e. the same number of field lines passes through any sphere centred at the origin regardless of its size and shape. Hence the flux through any surface enclosing the charge is equal to (q/ε_0).

If there are number of charges at the origin instead of a single charge, then according to the principle of superposition the total electric field is the sum of all the individual fields. Thus

$$E = \sum_{i}^{n} E_i$$

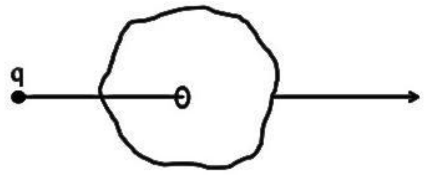


Figure 1.5

Then the flux through any surface enclosing all the fields is

$$\int_{Surface} E \cdot ds = \sum_{i}^{n} \left[\int_{Surface} E_i \cdot ds \right]$$

$$\int_{Surface} E \cdot ds = \sum_{i}^{n} \left[\frac{1}{\varepsilon_0} q_i \right]$$

As shown in Figure 1.5, a charge outside the surface would not contribute anything to the total flux, since its field lines enter in one side and emerge out the other side. Therefore for any closed surface, the total flux is

$$\int_{Surface} E \cdot ds = \left[\frac{1}{\varepsilon_0} Q_{Enclosed} \right] \qquad \text{--- 1.17}$$

Where $Q_{Enclosed} = q_1 + q_2 + q_3 + \cdot \ \cdot \ \cdot \ \cdot$ is the total charge enclosed within the surface. This is Gauss's law. The equation 1.17 is also known as the *'integral form of Gauss's law'*.

1.7.1 Differential form of Gauss's law

A differential form of Gauss's law for continuous charge distribution is obtained by applying the divergence theorem to equation 1.17

$$\int_{Surface} E \cdot ds = \int_{Volume} (\nabla \cdot E) \ d\tau \qquad \text{--- 1.18}$$

The charge enclosed within the surface S can be expressed in terms of the charge density ρas

$$Q_{Enclosed} = \int_{Volume} \rho \cdot d\tau$$

Now the Gauss's law (i.e. equation 1.17) becomes

$$\int_{Surface} E \cdot ds = \int_{Volume} \left(\frac{1}{\varepsilon_0} \rho \right) \cdot d\tau \qquad \text{--- 1.19}$$

Comparing equations 1.18 and 1.19, we get

$$\int_{Volume} (\nabla \cdot E) \ d\tau = \int_{Volume} \left(\frac{1}{\varepsilon_0} \rho \right) d\tau \qquad \text{--- 1.20}$$

The equation 1.20 is hold good for any integral and hence the integrands must be equal. Therefore

$$\nabla \cdot E = \left(\frac{1}{\varepsilon_0} \rho \right) \qquad \text{--- 1.21}$$

Equation 1.21represents the differential form of Gauss's law.It states that the divergence of the electric field intensity at any point is equal to ($1/\varepsilon_0$) times the volume charge density at that point. This is Maxwell's divergence form of electric field, E.

1.7.2 Applications of Gauss law

Gauss's law can be used tocalculate the electric field Efor symmetrical charge distribution, such as point charge, an infinite line charge, an infinite sheet of charge and a spherical distribution of charge.The symmetry is crucial for the application of Gauss's law in its integral form. There are only three kinds of symmetry. They are

(i) spherical symmetry,

(ii) cylindrical symmetry and

(iii) plane symmetry.

(a) Electric field due to a uniformly charged sphere or due to spherical charge distribution

Consider a uniform spherical charge distribution of radius R. Let q be the total charge on the sphere and ρbe the volume charge density. Now our aim is to find the electric filed intensity at a point P (i) outside the sphere, (ii) on the sphere and (iii) inside the sphere.

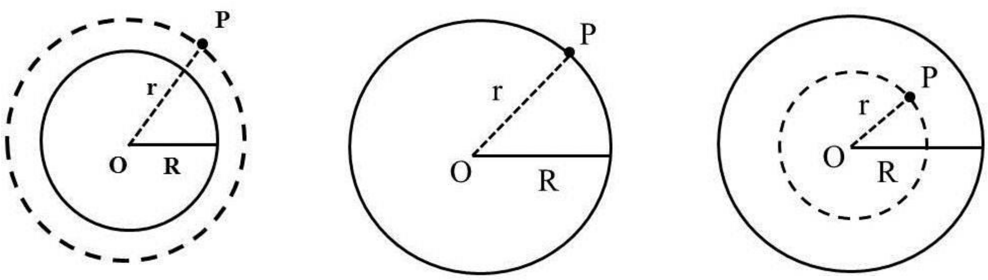
Case (i):***At a point outside the sphere***

Let us consider a point P at a distance r from the centre O, such thatr > R. To find the electric field at a point P, imagine a Gaussian spherical surface of radius r as illustrated in Figure 1.6a. By symmetry the E at all points on the Gaussian surface is equal and normal to the surface.The flux through the surface is given by

$$\int_{\text{Surface}} E \cdot ds = \int_{\text{Surface}} Eds \text{ (angle between E and ds is zero)}$$

Since E is constant over the Gaussian surface, then

$$E \int_{\text{Surface}} ds = E \cdot 4\pi r^2$$



(a) (b) (c)

Figure 1.6

By Gauss's law $$E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \text{ Newton/Coulomb} \quad \text{--- 1.22}$$

or $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \text{ Newton/Coulomb} \quad \text{--- 1.23}$$

Thus the electric field at any point outside the charged sphere is the same as if the charge on the sphere concentrated at the centre.

Case (ii) At a point on the surface of the sphere

Here r = R, (Figure 1.6b) hence

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \text{ Newton/Coulomb} \qquad \text{--- 1.24}$$

or $$\tilde{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r} \text{ Newton/Coulomb} \qquad \text{--- 1.25}$$

Case (iii)At apoint on the surface of the sphere

Let the point P be inside the sphere at a distance r from the centre O. Construct a Gaussian sphere with radius r passing through the point P as shown in Figure 1.6c. Then according to Gauss's law,

$$\int_{Surface} E \cdot ds = \frac{q}{\varepsilon_0}$$

In this case, the volume charge density $\rho = \frac{q}{\left(\frac{4}{3}\right)\pi R^3}$

But $$\int_{Surface} E \cdot ds = \frac{1}{\varepsilon_0}\rho\left(\frac{4}{3}\right)\pi r^3$$

$$E \int_{Surface} ds = \frac{1}{\varepsilon_0}\rho\left(\frac{4}{3}\right)\pi r^3$$

$$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\rho\left(\frac{4}{3}\right)\pi r^3$$

$$E = \frac{1}{4\pi r^2}\frac{1}{\varepsilon_0}\rho\left(\frac{4}{3}\right)\pi r^3$$

Substituting the value of ρ, we have

$$E = \frac{1}{4\pi r^2}\frac{1}{\varepsilon_0}\frac{q}{\left(\frac{4}{3}\right)\pi R^3}\left(\frac{4}{3}\right)\pi r^3$$

$$E = \frac{1}{4\pi}\frac{1}{\varepsilon_0}\frac{q}{\left(\frac{4}{3}\right)\pi R^3}\left(\frac{4}{3}\right)\pi r$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{4}{3}\right)\pi R^3}\left(\frac{4}{3}\right)\pi r$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3} = \frac{qr}{4\pi\varepsilon_0 R^3} \quad \text{--- 1.26}$$

or $$\tilde{E} = \frac{qr}{4\pi\varepsilon_0 R^3}\hat{r} \text{ Newton/Coulomb} \quad \text{--- 1.27}$$

Thus the electric field intensity at appoint inside the sphere depends on r. It is maximum atr=R.

(b) Electric field due to a uniformly charged cylinder

Consider a portion of a uniformly charged infinitely long cylinder with radius aand surface charge density σ. Now we wish to find the electric field at a point P which is at a distance r from the axis of the cylinder. For this construct a Gaussian surface in the form of a cylinder of length l with radius r and coaxial with the charged cylinder as shown in Figure 1.7.

The flux through the cylindrical Gaussian surface is

$$\int_{\text{Surface}} E \cdot ds = E\int ds = E2\pi rl$$

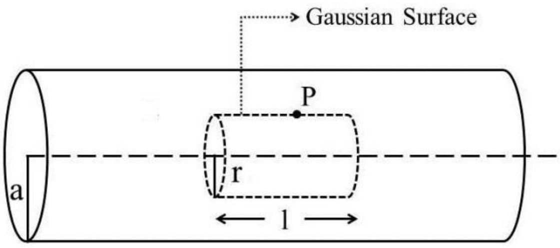


Figure 1.7

According to Gauss's law,

$$\int_{\text{Surface}} E \cdot ds = \left(\frac{1}{\varepsilon_0}\right) Q_{\text{enclosed}}$$

$$\int_{\text{Surface}} E \cdot ds = \frac{1}{\varepsilon_0} \text{ [Total charge enclosed within the Gaussian surface]}$$

The total charge enclosed by the Gaussian surface is given by

Charge = volume × volume density of charge

Hence the charge enclosed in Gaussian surface = $\pi r^2 l\rho$

$$\int_{\text{Surface}} E \cdot ds = \frac{1}{\varepsilon_0}\pi r^2 l\rho$$

$$E \int_{Surface} ds = \frac{1}{\varepsilon_0} \pi r^2 l \rho$$

$$E \cdot 2\pi r l = \frac{1}{\varepsilon_0} \pi r^2 l \rho$$

or $$E = \frac{r\rho}{2\varepsilon_0} \qquad \text{--- 1.28}$$

or $$\tilde{E} = \frac{r\rho}{2\varepsilon_0} \hat{r} \qquad \text{--- 1.29}$$

Thus the electric field E inside a charged cylinder is directly proportional to the distance of the point from the axis of the cylinder.

(c) Electric field due to an infinite plane of charge

Consider a portion of an infinite thin non-conducting plane sheet of charge as shown in Figure 1.8. To find the electric field E near it, construct a Gaussian surface of cross-sectional area A in the form of a pill-box extending equal distances above and below the planeas shown in Figure 1.8. The direction of the electric field E is normal to the end faces andaway from the plane. No lines of force pierce through the curved surface of the pill box, therefore the curved surface does not contribute to the flux.i.e. $\int_{Surface} E \cdot da = 0$. Hence the total flux is equal to the sum of the contribution from the two end faces. Thus

$$\int_{Surface} E \cdot ds = \left(\frac{1}{\varepsilon_0}\right) Q_{Enclosed}$$

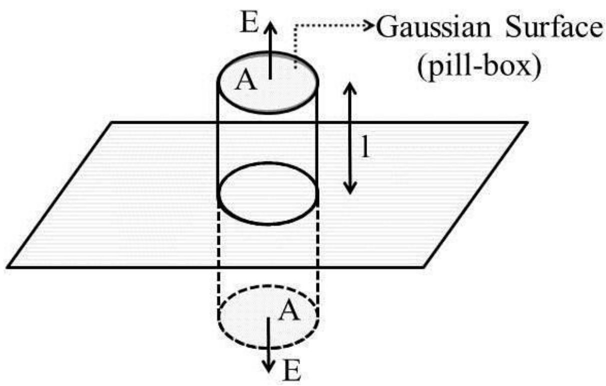


Figure 1.8

$$\int_{Surface} E \cdot ds = EA + EA = \left(\frac{1}{\varepsilon_0}\right) Q_{Enclosed}$$

$$\int_{Surface} E \cdot ds = 2EA = \left(\frac{1}{\varepsilon_0}\right) Q_{Enclosed}$$

The total charge enclosed by the cylinder is σA.

Hence

$$\int_{Surface} E \cdot ds = 2EA = \left(\frac{1}{\varepsilon_0}\right)\sigma A$$

$$2EA = \frac{\sigma A}{\varepsilon_0} \text{ or } E = \frac{\sigma A}{2A\varepsilon_0}$$

or $$E = \frac{\sigma}{2\varepsilon_0}$$

or $$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n} \quad \text{--- 1.30}$$

In equation 1.30, $\hat{n}$ is a unit vector pointing away from the surface. Here we note that the electric field is independent of the distance of the point from the sheet. So E is the same for all points on each side of the sheet.

1.8The potential function

The electric field around a charged body can be described not only by the electric field intensity E but also by a scalar quantity called the electric potential V.

The electric potential at a point in an electric field is defined as the amount of work done by an external agent in moving a unit positive charge from infinity to that point against the electrical force of the field. If W is the amount of work done in bringing a positive test charge Q from infinity to a point in the electric field, then the potential at that point is

$$V = \frac{W}{Q} \quad \text{--- 1.31}$$

If W is measured in joules, Q in Coulombs, then the potential V is in volts. Hence

$$1 \text{ Volt} = \frac{1 \text{ Joule}}{1 \text{ Coulomb}} \quad \text{--- 1.32}$$

Since both W and Q are scalar quantities, the potential V also a scalar.

1.8.1 The potential difference

Consider two points A and B separated by a distance r as shown in Figure 1.9. The potential difference between two points is numerically equal to the work done in moving a unit positive test charge (q) from one point (a) to another point (b).

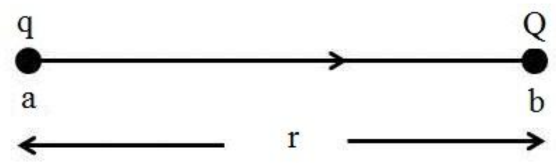


Figure 1.9

Thus the potential difference between two points a and b is defined as

$$V(b) - V(a) = \frac{W_{ab}}{q} \qquad \text{---1.33}$$

where, W_{ab} is the work done by an external agent in moving a positive test charge q from a to b.

1.8.2 Relation between V and E

Consider two points a and b in an electric field as shown in Figure 1.10. Let the charge q is moved from a to b along the path as shown by the external agent. The electric force exerted on q by Q is qE. The external agent would supply an equal and opposite force F on the charge in order to move it. Hence the force is

$$F = -qE \qquad \text{---1.34}$$

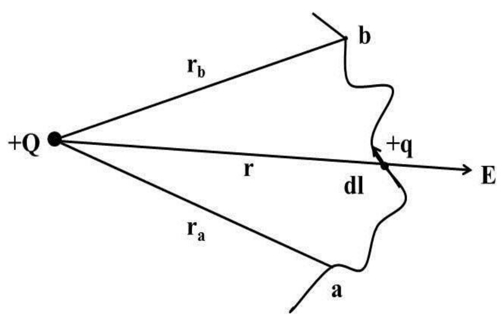


Figure 1.10

The charge moves a small distance dl along the closed path by the application of above force. Therefore the work done by the external agent is

$$dW = F \cdot dl \qquad \text{---1.35}$$

The total work done in moving the charge from a to b is obtained by integrating the equation 1.35.

$$\int dW = W_{ab} = \int_a^b F \cdot dl \qquad \text{---1.36}$$

Substituting equation 1.34 in 1.36, we have

$$W_{ab} = \int_a^b (-qE) \cdot dl$$

$$W_{ab} = -q\int_a^b E \cdot dl$$

$$\frac{W_{ab}}{q} = -\int_a^b E \cdot dl$$

But by definition

$$V(b) - V(a) = \frac{W_{ab}}{q} \quad \text{(From equation 1.33)}$$

Hence, the potential difference between the points a and b is

$$V(b) - V(a) = -\int_a^b E \cdot dl \qquad \text{--- 1.37}$$

From the fundamental theorem of gradient, we have

$$\int_a^b (\nabla f) \cdot dl = f(b) - f(a)$$

$$\int_a^b (\nabla V) \cdot dl = V(b) - V(a) \qquad \text{--- 1.38}$$

Now, using equation 1.37, the equation 1.38 can be written as

$$\int_a^b (\nabla V) \cdot dl = -\int_a^b E \cdot dl$$

This is true for any points a and b, therefore the integrands must be equal. Therefore

$$E = -\nabla V \qquad \text{--- 1.39}$$

Thus the electric field is the gradient of a scalar potential.

1.8.3 Potential due to a point charge

Consider a positive point charge Q. The electric field of Q is radially outwards as shown in Figure 1.11.

Let a test charge q be moved along a radial line from a to b to calculate the potential difference between them.

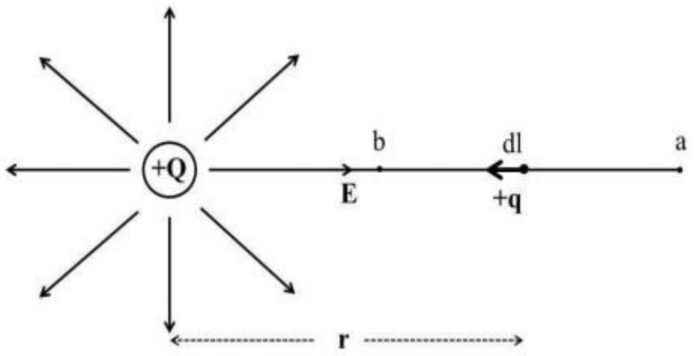


Figure 1.11

From equation 1.37, we have

$$V(b) - V(a) = -\int_a^b E \cdot dl \qquad \text{--- 1.37}$$

As E points in the right direction and test charge q is moved towards left i.e. b, E and dl are 180° apart. Therefore E.dl = –E.dl. Thus the eqn. 1.37 becomes

$$V(b) - V(a) = \int_a^b E \cdot dl$$

Since the distance r is measured from charge Q i.e. right and dl is measured towards left i.e. b, then dl = –dr. Hence the equation 1.30 becomes as

$$V(b) - V(a) = -\int_a^b E \cdot dr \qquad \text{--- 1.40}$$

The electric field of a point charge is (From equation 1.5)

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

Substituting in equation 1.40, we have

$$V(b) - V(a) = -\int_{r_a}^{r_b} \frac{q}{4\pi\varepsilon_0 r^2} \hat{r} dr$$

$$V(b) - V(a) = -\frac{q}{4\pi\varepsilon_0} \int_{r_a}^{r_b} \frac{1}{r^2} dr\, \hat{r}$$

$$V(b) - V(a) = -\frac{q}{4\pi\varepsilon_0} \left[\frac{1}{r_b} - \frac{1}{r_a} \right]$$

If the point a is situated at an infinity, then r = ∞. Hence V(a) = 0. Thus r_b = r, then the potential at b is represented by V.

$$V = \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{r} \right] \qquad \text{--- 1.41}$$

$$V = \frac{q}{4\pi\varepsilon_0 r} \qquad \text{--- 1.42}$$

1.9 Poisson's and Laplace equation

The electric field E can be written as the gradient of a scalar potential

$$E = -\nabla V \qquad \text{--- 1.39}$$

Taking divergence on both sides of equation 1.39, we get

$$\nabla \cdot E = -\nabla \cdot (\nabla V)$$

$$\nabla \cdot E = -\nabla^2 V \qquad \text{--- 1.43}$$

i.e. the divergence of E is the Lapalacian of V

The differential form of Gauss law is

$$\nabla \cdot E = \left(\frac{1}{\varepsilon_0} \rho \right) \qquad \text{--- 1.21}$$

Comparing eqns. 1.43 and 1.21, we have

$$\nabla^2 V = -\left(\frac{1}{\varepsilon_0} \rho \right) \qquad \text{--- 1.44}$$

This is known as Poisson's equation.

For Cartesian coordinate system

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z} \right)$$

$$\nabla \cdot \nabla V = \left(\frac{\partial^2 V}{\partial x^2}\right) + \left(\frac{\partial^2 V}{\partial y^2}\right) + \left(\frac{\partial^2 V}{\partial z^2}\right)$$

Poisson's equation for Cartesian coordinate system can be written as

$$\nabla^2 V = \left(\frac{\partial^2 V}{\partial x^2}\right) + \left(\frac{\partial^2 V}{\partial y^2}\right) + \left(\frac{\partial^2 V}{\partial z^2}\right) = -\left(\frac{1}{\varepsilon_0}\right)\rho \quad \text{--- } 1.45$$

If there are no charges in the region, i.e. for free space ρ=0. Hence equation 1.44 becomes

$$\nabla^2 V = -\left(\frac{0}{\varepsilon_0}\right) = 0 \quad \text{--- } 1.46$$

This is known as Laplacian equationand is valid only in the region of free charges.

From equation 1.45, we have

$$\nabla^2 V = \left(\frac{\partial^2 V}{\partial x^2}\right) + \left(\frac{\partial^2 V}{\partial y^2}\right) + \left(\frac{\partial^2 V}{\partial z^2}\right) = 0 \quad \text{--- } 1.47$$

This second order partial differential equation relating the rate of change of potential V in three space must be satisfied for any charge free region

An alternate of equation 1.47 is $\nabla \cdot E = 0$.

It means that the number of lines of electric field strength emerging from a unit volume is zero or lines of electric field strength are continuous.

Poisson's equation is a differential equation which relates the potential at a point to the volume charge density at that point. If the volume charge density in a region is zero, then Poisson's equation reduces to Laplace's equation. Laplace's equation states that the Laplacian of the electrostatic potential in a region devoid of charges is equal to zero,

If we take the curl of E, then

$$\nabla \times E = \nabla \times (-\nabla V) = 0$$

i.e. the curl of gradient is zero. The curl law is used only to show that the electric field E could be expressed as the gradient of a scalar.

1.10Equipotential surface

In an electric field, there are many points at which the electric potential is the same. There can be number of points which can be located at the same distance from the charge. The locus of the points, all of which have the same electric potential, is called an equipotential surface. In other words, an equipotential surface is an imaginary surface in an electric field of a given charge distribution, in which all the points on the surface are at the same electric potential.

The potential difference between any two points on the equipotential surface is always zero. It means the work done to move a test charge from one point to another in an equipotential surface is zero. For a particular charge distribution, there can be many equipotential surfaces existing in an electric field.

Consider a point charge sited at the origin of the sphere. The potential at any point which is at a distance r from the origin i.e. point charge is given by (From equation 1.41)

$$V = \frac{Q}{4\pi\varepsilon_0 r} \qquad \text{--- 1.41}$$

Therefore the potential is the same at all points which are at a distance r from Q and the surface joining all such points in the field is called an equipotential surface.

There exists other equipotential surfaces in the field of a point charge at $r=r_1$, $r=r_2$, ….. in the form of concentric spheres as shown in Figure 1.12.

It can benoted that the potential V is inversely proportional to the distance r. As a result V_1 at r_1 is the maximum and it goes on decreasing with the increase of r. Thus $V_1>V_2>V_3>$ ……However, the potential of equipotential surfaces goes on increasing as one move from infinity to the position of the charge i.e. against the direction of electric field.

In the case of uniform field, the equipotential surfaces are perpendicular to E and are equispaced for fixed increment of voltages.

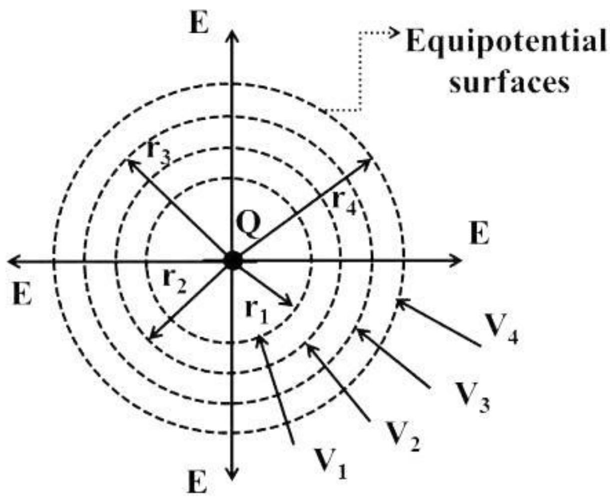


Figure 1.12

The work done to move a charge along a circular path of radius r_1 in the direction of dl as shown in Figure 1.13a is zero. This is because E and dl are perpendicular to each other. Thus the electric field and equipotential surfaces are mutually perpendicular.

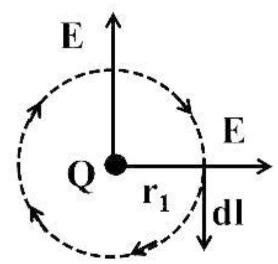


Figure 1.13a

However, in the case of non uniform field the field lines tend to diverge in the direction of decreasing E. Hence equipotential surfaces are still perpendicular to the direction of E but are not equispaced for fixed amount of increment voltages as shown in Figure 1.13b.

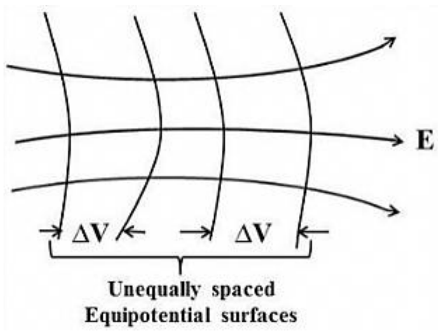


Figure 1.13b

1.11 Energy associated to an electrostatic field

Electrostatic energy may be considered as the energy obligatory to launch a given charge distribution in space. Let us calculate the amount of work required to assemble of point charges. For simplicity here we consider only three charges placed in free space as shown in Figure 1.14.

First consider the charge q_1. No work has to be done to bring the first charge q_1 at P_1, since there is no field initially to oppose the motion of charge q_1. However, work is needed to bring the charge q_2 nearer to q_1 at P_2. The work required to place the charge q_2 at P_2 is

$$W_2 = q_2 V_1 (P_2) \qquad \text{---1.48}$$

where V_1 is the potential due to charge q_1 at P_2.

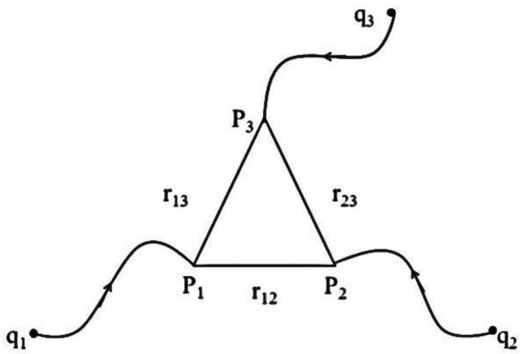


Figure 1.14

But the potential at any point in space is given by

$$V = \frac{q}{4\pi\varepsilon_0 r} \quad \text{--- 1.41}$$

Therefore

$$V_1(P_2) = \frac{q_1}{4\pi\varepsilon_0 r_{12}} \quad \text{--- 1.49}$$

where r_{12} is the distance between q_1 and q_2, when they are at P_1 and P_2 respectively. Substituting equation 1.48 in 1.49, we have

$$W_2 = q_2 \frac{q_1}{4\pi\varepsilon_0 r_{12}} \quad \text{---1.50}$$

or

$$W_2 = q_2 \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{12}}\right) \quad \text{--- 1.51}$$

Then the work required to bring q_3 is

$$W_3 = q_3 V_{1,2}(P_3) \quad \text{--- 1.52}$$

Where $V_{1,2}$ is the potential due to charges q_1 and q_2 at P_3

Therefore

$$V_{1,2}(P_3) = \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{q_1}{r_{13}}\right)+\left(\frac{q_2}{r_{23}}\right)\right] \quad \text{--- 1.53}$$

$$W_3 = q_3 \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{q_1}{r_{13}}\right)+\left(\frac{q_2}{r_{23}}\right)\right] \quad \text{--- 1.54}$$

Therefore the total work done to assemble the three charges is

$$W = W_1+W_2+W_3$$

$$W_2 = q_2 \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{12}}\right) + q_3 \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{q_1}{r_{13}}\right)+\left(\frac{q_2}{r_{23}}\right)\right]$$

$$W_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}}\right) + \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}}\right) + \frac{1}{4\pi\varepsilon_0}\left(\frac{q_2 q_3}{r_{23}}\right)$$

$$W_2 = \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{q_1 q_2}{r_{12}}\right)+\left(\frac{q_1 q_3}{r_{13}}\right)+\left(\frac{q_2 q_3}{r_{23}}\right)\right] \quad \text{--- 1.55}$$

In general

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\left[\left(\frac{q_i q_j}{r_{ij}}\right)\right] \quad \text{--- 1.56}$$

In equation 1.56, j>I indicates that not to count the same pair twice.

An alternate way to accomplish this is, count each pair twice and divide then by 2.

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\left[\left(\frac{q_i q_j}{r_{ij}}\right)\right]$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\left[\left(\sum_{\substack{j=1\\ j>i}}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}}\right)\right] \quad \text{--- 1.57}$$

The term within the parenthesis represents the potential at P_i, i.e. the position of q_i due to all charges.Thus

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \; V(P_i) \quad \text{--- 1.58}$$

The equation 1.58 represents the amount of energy stored in the configuration.

For a continuous charge distribution, we have for a volume v. Therefore the total charge can be represented by

$$q = \int_{\text{Volume}} \rho \, d\tau \quad \text{--- 1.59}$$

If V is the potential at the potential at the point occupied by the charge $\rho d\tau$, the electric potential energy of such a distribution is

$$W = \frac{1}{2}\int_{\text{Volume}} \rho V \, d\tau \quad \text{---1.60}$$

From differential form of Gauss law, we have

$$\nabla \cdot E = \left(\frac{1}{\varepsilon_0}\rho\right) \quad \text{---1.61}$$

or $$\rho = \varepsilon_0 (\nabla \cdot E) \qquad \text{--- 1.62}$$

Putting equation 1.62 in equation 1.60

$$W = \frac{1}{2} \int_{Volume} \varepsilon_0 (\nabla \cdot E) V \, d\tau$$

$$W = \frac{\varepsilon_0}{2} \int_{Volume} (\nabla \cdot E) V \, d\tau \qquad \text{--- 1.63}$$

From the vector product rule, we have

$$\nabla \cdot (EV) = V(\nabla \cdot E) + E \cdot (\nabla V)$$

$$(\nabla \cdot E)V = \nabla \cdot (EV) - E \cdot (\nabla V)$$

But E=-∇V, hence

$$(\nabla \cdot E)V = \nabla \cdot (EV) - E \cdot (-E)$$

$$(\nabla \cdot E)V = \nabla \cdot (EV) - E^2 \qquad \text{--- 1.64}$$

Putting equation 1.64 in 1.63, we obtain

$$W = \frac{\varepsilon_0}{2} \int_{Volume} \left[\nabla \cdot (EV) - E^2 \right] d\tau$$

$$W = \frac{\varepsilon_0}{2} \left[\int_{Volume} \nabla \cdot (EV) \, d\tau - \int_{Volume} E^2 d\tau \right]$$

By applying the divergence theorem to the first integral, we get

$$\int_{Volume} \nabla \cdot (EV) \, d\tau = \int_{Volume} (EV) \cdot ds$$

Thus the total work done is

$$W = \frac{\varepsilon_0}{2} \left[\int_{Volume} (EV) \cdot ds - \int_{Volume} E^2 d\tau \right] \qquad \text{--- 1.65}$$

Since the potential $V \; \alpha \frac{1}{r}$, field $E \; \alpha \frac{1}{r^2}$ and surface $ds \; \alpha \; r^2$, then the product EV.ds = $\frac{1}{r} \cdot \frac{1}{r^2} \cdot r^2 = \frac{1}{r}$. Thus if we pick larger and larger volumes, then the surface integral $\int_{Volume} (EV) \cdot ds$ becomes vanishingly small (because r becomes larger). Therefore the first term in equation 1.65 can be neglected and in particular the integration is taken over all space. Therefore

$$W = \frac{\varepsilon_0}{2} \int_{AllSpace} E^2 d\tau \qquad \text{--- 1.66}$$

1.12 Electrostatics Uniqueness Theorems

The boundary value problem can be solved by number of methods such as analytical, graphical, experimental, etc. Thus there is a question that, is the solution of Laplace's equation solved by any method, unique? The answer to this question is the uniqueness theorem.

Statement: Any solution to Laplace's or Poisson's equation which also satisfies the boundary condition must be the only solution that exists. It gives the uniqueness of the solution.

Consider a volume V surrounded by a surface S having charge density ρ. Let V_1 ad V_2 be the two potentials satisfying Poisson's equations as

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon} \quad \text{and} \quad \nabla^2 V_2 = -\frac{\rho}{\varepsilon} \qquad \text{--- 1.67}$$

Equation 1.67 suggest the formation of a difference potential as and it satisfies the Laplace's equation as

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 \qquad \text{--- 1.68}$$

Assume that V_1 and V_2 are also the two solutions of Laplace's equation. Both are the function of the coordinates of the system used. These solutions must satisfy Laplace's equation.

At the boundary, the potentials at different points are same due to equipotential surface. Then

$$V_1 = V_2 \qquad \text{--- 1.69}$$

Let V_0 be the difference between the solutions. Hence

$$V_1 - V_2 = V_0 \qquad \text{--- 1.70}$$

Using Laplace's equation for the difference V_0, we have

$$\nabla^2 V_0 = \nabla^2 (V_1 - V_2) = 0 \qquad \text{--- 1.71}$$

$$\nabla^2 V_1 - \nabla^2 V_2 = 0 \qquad \text{--- 1.72}$$

At the boundary $V_0 = 0$ (Using equations 1.69 and 1.70)

From Gauss divergence theorem, we have

$$\int_{\text{Volume}} (\nabla \cdot A)\, d\tau = \int_{\text{Surface}} A \cdot ds \qquad \text{--- 1.73}$$

Let $A = V_0 \nabla V_0$ and from vector algebra

$$\nabla \cdot (\alpha f) = \alpha (\nabla \cdot f) + f \cdot (\nabla \alpha)$$

Put $\alpha = V_0$ and $\nabla V_0 = f$. Hence we have

$$\nabla \cdot (V_0 \nabla V_0) = V_0(\nabla \cdot \nabla V_0) + \nabla V_0 \cdot (\nabla V_0)$$

But $\nabla \cdot \nabla = \nabla^2$

Therefore $\nabla \cdot (V_0 \nabla V_0) = V_0(\nabla^2 V_0) + \nabla V_0 \cdot \nabla V_0$ --- 1.74

Using equation 1.71, we write

$$\nabla \cdot (V_0 \nabla V_0) = V_0(0) + \nabla V_0 \cdot \nabla V_0 = \nabla V_0 \cdot \nabla V_0$$

$$\nabla \cdot (V_0 \nabla V_0) = \nabla \cdot (A) = \nabla V_0 \cdot \nabla V_0 \quad \text{--- 1.75}$$

Using equation 1.75, 1.73 can be written as

$$\int_{\text{Volume}} \nabla \cdot (V_0 \nabla V_0)\, d\tau = \int_{\text{Surface}} (V_0 \nabla V_0) \cdot ds \quad \text{--- 1.76}$$

$$\int_{\text{Volume}} (\nabla V_0 \cdot \nabla V_0)\, d\tau = \int_{\text{Surface}} (V_0 \nabla V_0) \cdot ds \quad \text{--- 1.77}$$

Since $V_0 = 0$ at the boundary, the right hand side of equation 1.77 is zero.

$$\int_{\text{Volume}} (\nabla V_0 \cdot \nabla V_0)\, d\tau = 0$$

This is the volume integral to be evaluated on the volume enclosed by the boundary. It is well know that $Y \cdot Y = |Y|^2$.

$$\int_{\text{Volume}} |\nabla V_0|^2\, d\tau = 0 \quad \text{as } \nabla V_0 \text{ is a vector}$$

The integration can be zero under two conditions

(i) The quantity under integral sign is zero.

(ii) The quantity is positive in some regions and negative in some regions by equal amount and hence zero.

But square term cannot be negative in any region hence, quantity under integral must be zero.

$$|\nabla V_0|^2 = 0$$

$$\nabla V_0{}^2 = 0$$

As the gradient of $V_0 = V_1 - V_2 = 0$ means $V_1 - V_2$ is constant and not changing with any coordinates. But at the boundary, it can be proved that $V_1 - V_2$ = constant = 0. Therefore

$$V_1 = V_2$$

This proves that both the solutions are equal and can not be different.

Thus the uniqueness theorem can be stated that if the solution of Laplace's equation satisfies the boundary condition then the solution is uniqueirrespective of the method it is obtained.Also the solution of Laplace's equation provides the field which is unique, satisfying the same boundary conditions in a given region.

UNIT II - MAGNETOSTATICS

Lorentz force – Faraday's law – Magnetic field strength and Ampere's circuital law – Biot-Savart's law – Ampere's force law – Magnetic vector potential – Equation of continuity – The far magnetic field of a current distribution – Magnetic field due to volume distribution of current.

2.1. Introduction

The electrostatics field exist due to charges at rest. The magnetic field exists due to a permanent magnet, motion of charges or current elements.

When the charges are in motion, they are surrounded by a magnetic field. The flow of charges constitutes an electric current. Thus a current carrying conductor is always surrounded by a magnetic field. If a current flow is steady then the magnetic field produced is also a steady magnetic field. The direct current (d.c.) is a steady flow of current hence the magnetic field produced by a conductor carrying d.c. current is a steady magnetic field. The study of steady magnetic field produced due to the flow of direct current through a conductor is called "magnetostatics".

2.2 Lorentz force

In a given magnetic field, the magnitude of the force on a charge is proportional to the magnitude of the charge Q and to the speed v of the charge. It has also been found to be proportional to the sine of the angle between the velocity v and the magnetic field induction B.

Therefore

$$F_{Magnetic} \quad \alpha \quad Q\,(v \times B) \qquad \text{--- 2.1}$$

$$F_{Magnetic} = K\,Q\,(v \times B)$$

In S.I. units, K=1, hence

$$F_{Magnetic} = Q\,(v \times B) \qquad \text{--- 2.2}$$

The force experienced by a charge Q in an electric field, E, is

$$F_{Electric} = Q\,E$$

If both electric and magnetic fields are present in a region, then the force experienced by a charge Q moving with a velocity v is given by the sum of the electric and magnetic forces i.e.

$$F_{Total} = F = F_{Magnetic} + F_{Electric} \qquad \text{--- 2.3}$$

$$F = Q\ (v \times B) + QE$$

$$F = Q\ [(v \times B) + E] \quad \text{--- 2.4}$$

or

$$F = Q\ [E + (v \times B)] \quad \text{--- 2.5}$$

This force is known as Lorentz's force law or Lorentz's force equation and the force given by it is known as the Lorentz's force.

2.3 Faraday's law

The space around a magnet yields the magnetic field B. The total magnetic lines of force across a given area is called the magnetic flux ϕ. When a magnet is introduced into the coil, each turn of the coil cuts the flux lines. The rate of cutting these lines i.e. $(d\phi/dt)$ depends on the speed of introducing the magnet. If the coil and the magnet are at rest then $(d\phi/dt) = 0$, even though the magnetic lines are linked with the coil. According to Faraday, the magnitude of induced emf in a coil is equal to $(d\phi/dt)$.

It is an experimental observation that an emf appears in a circuit when the magnetic flux through the circuit changes from any cause. Whenever the magnetic flux linked through a closed conductor is changed, an emf is induced in the conductor. The magnitude of this induced emf is equal to the negative time rate of change of magnetic flux through the circuit. i.e.

$$e = -\frac{d\phi}{dt}$$

The emf induced in a loop of wire moving in the presence of a magnetic field is given by the flux rule.

$$e = -\frac{d\phi}{dt} \quad \text{--- 2.6}$$

In this case the magnetic force sets up the emf when the stationary loop moves in the magnetic field. Keeping the loop as stationary and if the magnet is moved then the magnetic flux in the neighbourhood of the loop is changed. This change in magnetic flux induces an electric field and the emf is the same as above. Thus, we can write

$$\int E \cdot dl = e = -\frac{d\phi}{dt} \quad \text{--- 2.7}$$

where E is the electric field.

Equation 2.7 is Faraday's law in integral form. Stoke's theorem is used to convert it into differential form.

$$\int \mathrm{E} \cdot \mathrm{dl} = \int_{\text{Surface}} (\nabla \times \mathrm{E}) \cdot \mathrm{ds} \qquad \text{--- 2.8}$$

Comparing eqns. 2.7 and 2.8, we have

$$\int_{\text{Surface}} (\nabla \times \mathrm{E}) \mathrm{ds} = -\frac{\mathrm{d}\phi}{\mathrm{dt}} \qquad \text{--- 2.9}$$

According to Gauss's law, the total electric flux passing through the surface is equal to the surface integral of magnetic field density over the surface.

$$\phi = \int_{\text{Surface}} \mathrm{B} \cdot \mathrm{ds}$$

Substituting the value of ϕ in equation 2.9, we get

$$\int_{\text{Surface}} (\nabla \times \mathrm{E})\, \mathrm{ds} = -\frac{\mathrm{d}\phi}{\mathrm{dt}} = -\frac{\mathrm{d}}{\mathrm{dt}} \int_{\text{Surface}} \mathrm{B} \cdot \mathrm{d} \qquad \text{--- 2.10}$$

$$\int_{\text{Surface}} (\nabla \times \mathrm{E})\, \mathrm{ds} = \int_{\text{Surface}} \frac{\partial \mathrm{B}}{\partial \mathrm{t}} \cdot \mathrm{ds}$$

or

$$(\nabla \times \mathrm{E}) = -\frac{\partial \mathrm{B}}{\partial \mathrm{t}} \qquad \text{--- 2.11}$$

The above equation 2.11 relates the space derivatives of E at a particular point to the time rate of change of B at the same point.

2.4 Magnetic field strength and Ampere circuital law

Amagnetic field at any point is characterized by another physical quantity called intensity of the magnetic field (H). The quantitative measure of strongness or weakness of the magnetic field is given by the magnetic field intensity or magnetic field strength. The magnetic field strength at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point. Consider a current carrying wire as shown in Figure 2.1. Now it is possible to determine B at all points in a region about a long current carrying wire.

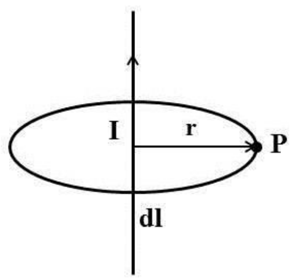


Figure 2.1

For a homogenous medium, the amount of magnetization B due to a current carrying wire is (i) proportional to the permeability (μ) of the medium (i.e. B α μ), (ii) proportional to the current I flowing (B α I) and (iii) inversely proportional to the distance r from the current carrying wire $\left(B \alpha \frac{1}{r}\right)$. Therefore

$$B \alpha \frac{\mu I}{r} \qquad \text{--- 2.12}$$

Where $\mu = \mu_r \mu_0$ (μ_r =is the relative permeability)

$$B = K\frac{\mu I}{r} \qquad \text{--- 2.13}$$

$$B = \frac{\mu I}{2\pi r}\left[\because K = \frac{1}{2\pi}\right] \qquad \text{---2.14}$$

In other words, the magnetic field intensity or strength of the magnetic field H at a point is defined as the ratio between the magnetic field B and the permeability of the surrounding medium. Thus

$$H = \frac{B}{\mu} \qquad \text{--- 2.15}$$

The magnetic flux lines are measured in weber while magnetic field intensity is measured in Newton/weber or amperes/metre. It is a vector quantity. This is similar to the electric field intensity E in electrostatics.

From equation 2.15

$$B = \mu H \qquad \text{--- 2.16}$$

Equating equations 2.14 and 2.16

$$\frac{\mu I}{2\pi r} = \mu H$$

$$H = \frac{I}{2\pi r}\,\text{Amp/m} \qquad \text{--- 2.17}$$

The line integral $F_{Magnetic} = \int_a^b H \cdot dl$ is defined as the magnetomotive force between the points a to b.

For a circular path of radius r, we have L=2πr, hence

$$F_{Magnetic} = \int_a^b H \cdot dl = H \cdot L = H \cdot 2\pi r = I \ [\text{Using equation 2.17}]$$

This is the Ampere's work law or circuital law.

Statement: The line integral of the tangential component of magnetic field strength around a closed path is equal to the current enclosed by the path i.e.

$$\int H \cdot dl = I_{Enclosed} \qquad \text{--- 2.18}$$

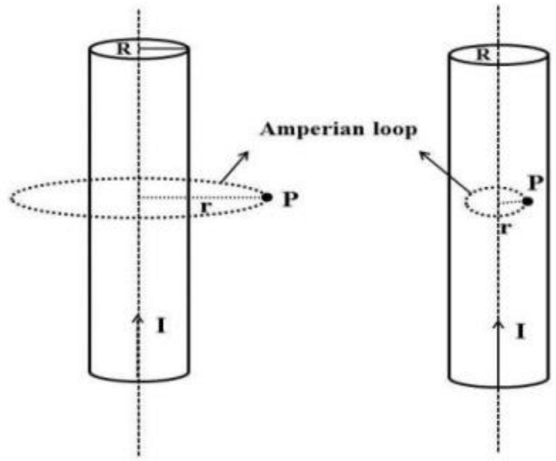
or

$$\int B \cdot dl = \mu_o I_{Enclosed} \qquad \text{--- 2.19}$$

2.4.1 Applications of Ampere's law

(a) The magnetic field due to a long current carrying conductor

Let I be the current through the conductor of radius R. Consider an amperian loop of radius r as shown in Figure 2.2. The magnitude of B around an amperian loop of radius r is constant centred at the conductor. The lines of B are concentric circle and the direction is circumferential.



(a) (b)

Figure 2.2

Hence the field B at a point P which is at a distance r form the axis of wire (Figure 2.2) is given by Ampere's circuital law as

$$\int B \cdot dl = \mu_0 \int J \cdot ds$$

$$\int B \cdot dl = \mu_0 I_{Enclosed} \qquad \text{--- 2.20}$$

$$\int B \cdot dl = \mu_0 \text{ [Current crossing the bounded surface]}$$

Special cases:

Case (i) If r > R, The current crossing bounded surface is I. Then

$$\int B \cdot dl = B\int dl = B \cdot 2\pi r = \mu_0 I_{Enclosed} = \mu_0 I$$

$$B \cdot 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r} \quad \text{--- 2.21}$$

Case (ii) If r < R, the current crossing the bounded surface will be

$$I_{Ennclosed} = J \times \pi r^2 = \left(\frac{I}{\pi R^2}\right) \times \pi r^2 = I\frac{r^2}{R^2}$$

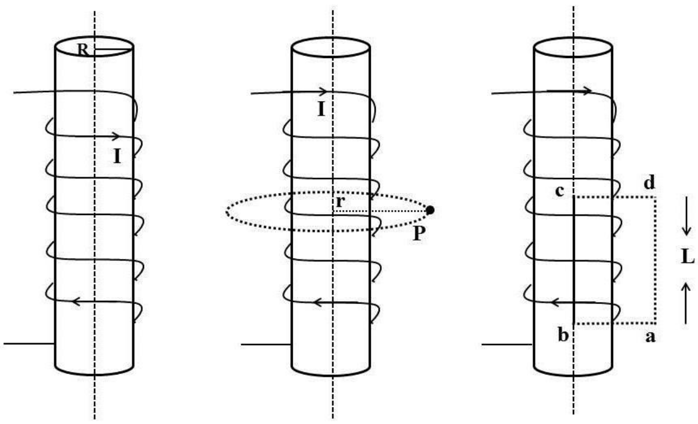
$$\int B \cdot dl = B\int dl = B \cdot 2\pi r = \mu_0 I \frac{r^2}{R^2}$$

$$B = \mu_0 I \frac{r^2}{R^2}\frac{1}{2\pi r}$$

$$B = \frac{\mu_0 I r}{2\pi R^2} \quad \text{--- 2.22}$$

(b) The magnetic field due to a long solenoid

Consider a long solenoid consisting of N number of closely wound turns per unit length on a cylinder of radius R and carrying current I as shown in Figure 2.3a.



(a) (b) (c)

Figure 2.3

To determine the value of B outside the solenoid, consider the amperian loop of radius r as shown in Figure 2.3b. Hence

$$\int B \cdot dl = B \cdot 2\pi r = \mu_0 I_{Ennclosed} = 0$$

This is because the loop enclosed no current. Therefore the value of B outside the solenoid is zero.

To determine the value of B inside the solenoid, consider the rectangular shaped amperian loop of height L which is half inside and half outside as shown in Figure 2.3c.

Now let us apply Ampere's law $\int B \cdot dl = \mu_0 I_{Ennclosed}$ to the path 'abcd' as in Figure 2.3c.

Here the total integral is the sum of four integrals one for each path of rectangle.

$$\text{i.e.} \quad \int_{abcd} B \cdot dl = \int_a^d B \cdot dl + \int_d^c B \cdot dl + \int_c^b B \cdot dl + \int_b^a B \cdot dl$$

For the path ba, B and dl are perpendicular to each other and hence B·dl = 0 and similarly for dc, B·dl = 0. The path ad is outside the solenoid. Therefore B·dl = 0, because B is zero outside.

The only contribution to the field is due to the path bc. Thus for entire path

$$\int_{abcd} B \cdot dl = \int_c^b B \cdot dl$$

B and dl are parallel inside, hence

$$\int_{abcd} B \cdot dl = B \int_c^b dl$$

But $\int_c^b dl$ = length bc = L. Thus

$$\int_{abcd} B \cdot dl = BL = \mu_0 I_{Ennclosed}$$

The net current $I_{Ennclosed}$ passes through the area bounded by the path of integration is not the same as the current I in the solenoid because the path of integration encloses more than one turn.

Let N be the number of turns per unit length, then

$$I_{Ennclosed} = INL$$

$$\int_{abcd} B \cdot dl = BL = \mu_0 I_{Ennclosed} = \mu_0 INL$$

$$BL = \mu_0 INL$$

$$B = \mu_0 IN \qquad \text{--- 2.23}$$

(c) The magnetic field of a toroid coil

Toroid is a solenoid bent around in the form of a closed ring (Figure 2.4). If N is the number of turns in the toroid and I is the current in each turn.

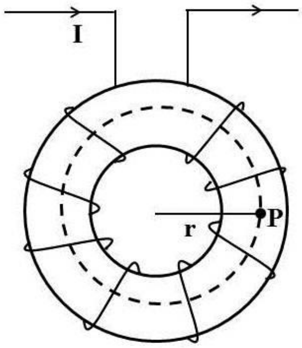


Figure 2.4

Now let us apply Ampere's law to a circle of radius r about the axis of toroid. Thus

$$\int_{abcd} B \cdot dl = B \cdot 2\pi r = \mu_0 I_{Enclosed} = \mu_0 In$$

Where n is the total number of turns in the toroid

$$B = \frac{\mu_0 nI}{2\pi r} = \mu_0 I\left(\frac{n}{2\pi r}\right)$$

$$B = \mu_0 IN \qquad \text{--- 2.24}$$

where $N = \left(\frac{n}{2\pi r}\right)$, the number of turns per unit length. The expression 2.24 is the same as for a long solenoid. The field outside the toroid coil is zero.

2.5 Ampere's Force law

Ampere observed that the force between the two current elements dl_1 and dl_2 carrying currents I_1 and I_2 respectively and separated by a distance r depends upon the following facts

(i) the force varies directly as the product of magnitude of currents

(ii) the force varies inversely as the square of the distance between the two current elements

(iii) the force depends upon the nature of the medium and

(iv) the force depends upon the lengths and orientations of the two current elements.

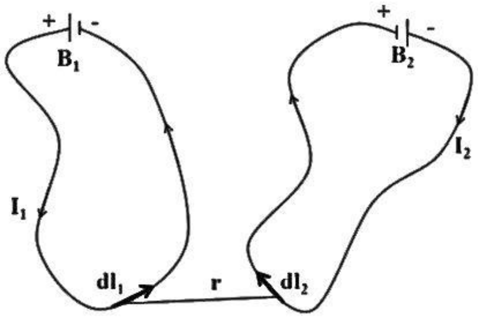


Figure 2.5

Consider the two current elements as shown in Figure 2.5. The force exerted on current element dl_2 by the current element dl_1 in free space is given by

$$dF_{21} = \left(\frac{\mu_0}{4\pi}\right)(I_1 I_2)\left(\frac{1}{r^2}\right)\left[dl_2 \times (dl_1 \times \hat{r})\right] \qquad \text{--- 2.25}$$

Where $\hat{r}$ represents the unit vector along the direction of r.

Since the conductor is made up of all such differential elements, then the expression for dF_{21} can be put in the form for the whole length of the conductors and is obtained by taking the integral of equation2.25. Therefore

$$F_{21} = \left(\frac{\mu_0}{4\pi}\right)\int_1\int_2 \frac{I_2 dl_2 \times (I_1\ dl_1 \times \hat{r})}{r^2} \qquad \text{--- 2.26}$$

The equation 2.26 represents the force exerted on current I_2 by current I_1. The line integral is required to ensure that all the current elements are considered. This is because current can flow only in the closed path, provided by the circuit.

The vectors dl_1 and dl_2 point in the direction of positive current flow. Force is measured in Newton, current in Ampere and length in metre.

In order to determine the direction of force, we first find the cross product $dl_1 \times \hat{r}$ and then obtain the cross product of dl_2 with $dl_1 \times \hat{r}$.

In equation 2.26, we can take I_1 and I_2 outside the integrals, so that

$$F_{21} = \left(\frac{\mu_0}{4\pi}\right) I_1\ I_2 \int_1\int_2 \frac{dl_2 \times (dl_1 \times \hat{r})}{r^2} \qquad \text{--- 2.27}$$

where integrals are taken around the two loops and the constant μ_0 is called the permittivity of free space and its value is equal to $4\pi \times 10^{-7}$ N/amp^2.

Equation 2.27 is the Mathematical statement of Ampere's observations about forces between current carrying loops and is called Ampere's law of force.

Equation 2.27 can be written in a more practical form as

$$F_{21} = I_2 \int_2 dl_2 \times \left(\frac{\mu_0}{4\pi} I_1 \int_1 \frac{(dl_1 \times \hat{r})}{r^2} \right) \quad \text{--- 2.28}$$

$$F_{21} = I_2 \int_2 dl_2 \times B_{21} \quad \text{--- 2.29}$$

where

$$B_{21} = \left(\frac{\mu_0}{4\pi} I_1 \int_1 \frac{(dl_1 \times \hat{r})}{r^2} \right) \quad \text{--- 2.30}$$

can be considered to be the magnetic field of circuit 1 at the position of dl_2 of circuit 2. The vector B is called the magnetic induction or the magnetic flux density and is expressed in Weber/metre^2 or Tesla.

2.6 Biot-Savart law

Biot-Savart law states that the magnetic field intensity dH produced at a point P as shown in Figure 2.6, by the differential current element Idl is proportional to the product Idl and the sine of the angle β between the element and the line joining P to the element dl and is inversely proportional to the square of the distance r between P and the element.

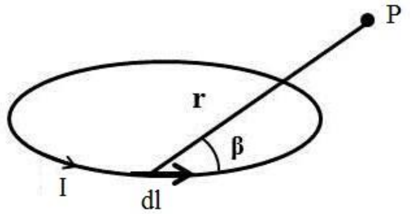


Figure 2.6

That is

$$dH \; \alpha \; \frac{Idl \sin\beta}{r^2} \quad \text{--- 2.31}$$

$$dH = K \frac{Idl \sin\beta}{r^2}$$

$$dH = \left(\frac{1}{4\pi}\right) \frac{Idl \sin\beta}{r^2} \quad \left[\because K = \frac{1}{4\pi}\right]$$

$$dH = \frac{Idl \sin\beta}{4\pi r^2}$$

More conveniently in vector form $dH = \frac{Idl \times \hat{r}}{4\pi r^2}$ --- 2.32

The magnetic field intensity at P due to the whole circuit is obtained by integrating dH. Hence

$$\int dH = \int \frac{Idl \times \hat{r}}{4\pi r^2}$$

$$H = \frac{1}{4\pi} \int \frac{I \times \hat{r}}{r^2} dl$$

or $$B = \frac{\mu_0}{4\pi} \int \frac{I \times \hat{r}}{r^2} dl \quad \text{--- 2.33}$$

From equation 2.33, it is evident that the magnetic field induction B at a position P due to current carrying element dl (Figure 2.4) will be equal to

$$B = \frac{\mu_0}{4\pi} \int I \left(\frac{dl \times \hat{r}}{r^2} \right) \quad \text{--- 2.34}$$

The equation 2.34 for B is called the Biot-Savart's law for line current.

2.6.1 Biot-Savart's Law for volume and surface currents

If the current I is distributed in space with a current density J amp/metre^2, then

$$I = J \cdot ds$$

$$I \cdot dl = J \cdot ds\, dl \ [\because ds\, dl = d\tau]$$

$$I \cdot dl = J \cdot d\tau$$

Hence the Biot-Savart's law for volume current is

$$B = \frac{\mu_0}{4\pi} \int \left(\frac{J \times \hat{r}}{r^2} d\tau \right) \quad \text{--- 2.35}$$

where the integration is carried out over any volume which includes all currents.

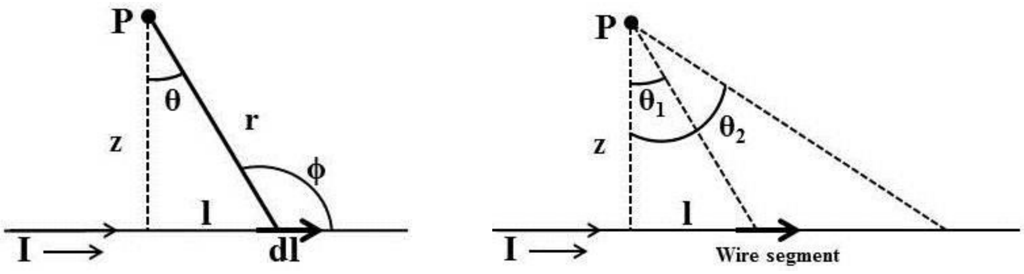
The Biot-Savart's law for surface current is given by

$$B = \frac{\mu_0}{4\pi} \int \left(\frac{K \times \hat{r}}{r^2} ds \right) \quad \text{--- 2.36}$$

2.6.2 Application of Biot-Savart's law

(i) The magnetic field B at a distance z above a long straight wire carrying a steady current I.

Consider an infinitely long current carrying wire as shown in Figure 2.7. Our aim is to determine the magnetic field at a point P.



(a) (b)

Figure 2.7

In Figure 2.7, we have $dl \times \hat{r} = dl \ \sin\phi$

$$dl \times \hat{r} = dl \ \sin\ (90 + \theta)$$

$$dl \times \hat{r} = dl \ \cos\theta$$

and $\tan\theta = \frac{l}{z}$

$$l = z \ \tan\theta$$

$$dl = z \frac{1}{\cos^2\theta} d\theta$$

Also we have $\cos\theta = \frac{z}{r}$

$$\frac{1}{r} = \frac{\cos\theta}{z} \text{ and}$$

$$\frac{1}{r^2} = \frac{\cos^2\theta}{z^2}$$

The field of any straight segment of wire in terms of the initial and final angles θ_1 and θ_2 respectively for the Figure 2.7b is obtained from the equation 2.34. Thus

$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} z \ \frac{1}{\cos^2\theta} d\theta \cos\theta \ \frac{\cos^2\theta}{z^2} \qquad \text{--- 2.37}$$

$$B = \frac{\mu_0 I}{4\pi \ z} \int_{\theta_1}^{\theta_2} \cos\theta \ d\theta$$

$$B = \frac{\mu_0 I}{4\pi \ z} [\sin\theta_2 - \sin\theta_1]$$

If the length of the wire is infinite one, then we have $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$

$$B = \frac{\mu_0 I}{4\pi\ z}\left[\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right]$$

$$B = \frac{\mu_0 I}{4\pi\ z}\left[1-(-1)\right]$$

$$B = \frac{\mu_0 I}{4\pi\ z}\left[2\right]$$

or $$B = \frac{\mu_0 I}{2\pi\ z} \qquad \text{--- 2.38}$$

Thus the magnetic field at a distance z form the infinitely long wire is directly proportional to the amount of current flowing through the wire, the medium in which the experiment is carried out and inversely proportional to the distance of the point P.

2.7 Magnetic vector potential

In electrostatics, the electric field intensity can be derived from the potential V by the relation $E = -\nabla V$. Here we show that the magnetic induction B can be related to a quantity A by the relation $B = \nabla \times A$, where A is called the magnetic vector potential.

According to Biot-Savart's law the magnetic induction vector B for line current is given by

$$B = \frac{\mu_0}{4\pi}\int I\frac{dl \times r}{r^2}$$

Also $$\frac{\hat{r}}{r^2} = -\nabla\left(\frac{1}{r}\right)$$

Then we write

$$B = -\frac{\mu_0}{4\pi}\int I\ dl \times \nabla\left(\frac{1}{r}\right) \qquad \text{--- 2.39}$$

$$B = \frac{\mu_0}{4\pi}\int\ \nabla\left(\frac{1}{r}\right) \times Idl$$

$$B = \frac{\mu_0}{4\pi} I\int \nabla\left(\frac{1}{r}\right) \times dl \qquad \text{--- 2.40}$$

But we know that $\nabla \times (fA) = f(\nabla \times A) - A \times \nabla(f)$ and putting dl for A and $\frac{1}{r}$ for f, we have

$$\nabla \times \left(\frac{1}{r}dl\right) = \frac{1}{r}(\nabla \times dl) - dl \times \left(\nabla\frac{1}{r}\right) \qquad \text{--- 2.41}$$

$$dl \times \left(\nabla\frac{1}{r}\right) = \frac{1}{r}(\nabla \times dl) - \nabla \times \left(\frac{1}{r}dl\right)$$

$$dl \times \nabla\left(\frac{1}{r}\right) = \frac{1}{r}(\nabla \times dl) - \nabla \times \left(\frac{1}{r}dl\right)$$

$$\nabla\left(\frac{1}{r}\right) \times dl = -\frac{1}{r}(\nabla \times dl) + \nabla \times \left(\frac{1}{r}dl\right)$$

$$\nabla\left(\frac{1}{r}\right) \times dl = \nabla \times \left(\frac{1}{r}dl\right) - \frac{1}{r}(\nabla \times dl)$$

Now the equation 2.40 becomes as

$$B = \frac{\mu_o}{4\pi} I \int \left[\nabla \times \left[\frac{1}{r}\right] dl - \frac{1}{r}(\nabla \times dl) \right]$$

$$B = \frac{\mu_0}{4\pi} I \int \nabla \times \left(\frac{1}{r}dl\right) - \int \frac{1}{r}(\nabla \times dl)$$

But $(\nabla \times dl)$=0, because dl does not depend on x,y,z. Thus

$$B = \frac{\mu_0}{4\pi} I \int \left(\nabla \times \frac{dl}{r} \right)$$

Interchanging the operators, we get

$$B = \frac{\mu_0 I}{4\pi} \nabla \times \left(\int \frac{dl}{r} \right) \qquad \text{--- 2.42}$$

$$B = \nabla \times \left(\frac{\mu_0 I}{4\pi} \int \frac{dl}{r} \right)$$

$$B = \nabla \times A \qquad \text{--- 2.43}$$

Where

$$A = \frac{\mu_0 I}{4\pi} \int \frac{dl}{r} \qquad \text{--- 2.44}$$

and is called the magnetic vector potential. Therefore the magnetic field induction, B, is given by the curl of vector potential.

Special cases:

Case (i) It satisfies the Poisson's equation.

If the current is distributed with a current density J, then I = J · da. Putting da·dl=dτ and I·dl = J da·dl = Jdτ. Integrating over the whole volume, we get from equation 2.44

$$A = \frac{\mu_0}{4\pi} \int \frac{J}{r} d\tau \qquad \text{--- 2.45}$$

Then

$$\nabla^2 A = \nabla^2 \left[\frac{\mu_0}{4\pi} \int \frac{J}{r} d\tau \right]$$

$$\nabla^2 A = \frac{\mu_0}{4\pi}\int \nabla^2\left(\frac{J}{r}\right)d\tau$$

$$\nabla^2 A = \frac{\mu_0}{4\pi}\int J\ \nabla^2\left(\frac{1}{r}\right)d\tau$$

$$\nabla^2 A = \frac{\mu_0 J}{4\pi}\int \nabla^2\left(\frac{1}{r}\right)d\tau$$

This is because J is not a function of x, y, z. But $\int \nabla^2\left(\frac{1}{r}\right)d\tau = -4\pi$.

$$\nabla^2 A = \frac{\mu_0}{4\pi}J(-4\pi)$$

$$\nabla^2 A = -\mu_0 J \qquad \text{--- 2.46}$$

Case (ii) The line integral of magnetic vector A around a closed path gives the magnetic flux linked with the area enclosed by the closed path. i.e.

$$\phi_B = \int A \cdot dl \qquad \text{--- 2.47}$$

By definition, we have $\phi_B = \int B \cdot ds = \int (\nabla \times A)\cdot da$

But according to Stoke's theorem, we have

$$\int_S (\nabla \times A)\cdot da = \int A \cdot dl$$

$$\phi_B = \int A \cdot dl$$

Case (iii) The divergence of magnetic vector potential A is zero or a scalar constant.

By Definition, we have $B = \nabla \times A$

Then

$$\nabla \times B = \nabla \times \nabla \times A$$

$$\nabla \times B = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\text{But } \nabla \times B = \mu_0 J \text{ and } \nabla^2 A = -\mu_0 J$$

Therefore

$$\mu_0 J = \nabla(\nabla \cdot A) - (-\mu_0 J)$$

$$\mu_0 J = \nabla(\nabla \cdot A) + (\mu_0 J)$$

$$\nabla(\nabla \cdot A) = \mu_0 J - (\mu_0 J)$$

$$\nabla(\nabla \cdot A) = 0 \qquad \text{--- 2.48}$$

This means that the divergence of A is zero i.e. $\nabla \cdot A = 0$

2.8 The far magnetic field of a current distribution

The concept of magnetic vector potential (i.e. equation 2.44) is valid only for any observation point. But if the observation point is far away from the source potential responsible for current distribution, then approximation is required for practical condition. Consider a current carrying loop of wire as shown in Figure 2.8.

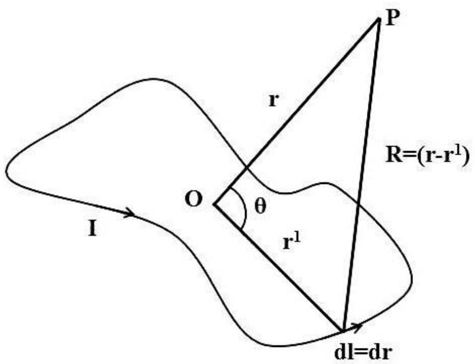


Figure 2.8

For a given current distribution as shown in Figure 2.8, the vector potential of a current loop is given by

$$A = \frac{\mu_0 I}{4\pi}\int \frac{dl}{R} \qquad \text{--- 2.49}$$

The point P and the current element dl are at a distances of r and r^1 respectively from the origin O and θ is the angle between r and r^1.

From the Figure 2.8, we have

$$R^{-1} = \frac{1}{R} = \frac{1}{(r - r^1)} = (r - r^1)^{-1}$$

$$(r - r^1)^{-1} = (r^2 - 2r.r^1 - r^{1^2})^{-\frac{1}{2}}$$

$$(r - r^1)^{-1} = \left[r^2 (1 - \frac{2r.r^1}{r^2} - \frac{r^{1^2}}{r^2}) \right]^{-\frac{1}{2}}$$

$$(r - r^1)^{-1} = \left[\left(r^2\right)^{\frac{-1}{2}} \left\{ 1 - \frac{2r.r^1}{r^2} - \frac{r^{1^2}}{r^2} \right\}^{-\frac{1}{2}} \right]$$

$$(r - r^1)^{-1} = \left[\frac{1}{(r^2)^{\frac{1}{2}}} \left\{ 1 - \frac{2r.r^1}{r^2} - \frac{r^{1^2}}{r^2} \right\}^{-\frac{1}{2}} \right]$$

$$(r - r^1)^{-1} = \frac{1}{r} \left[1 - \frac{2r\,r^1 \cos\theta}{r^2} - \frac{r^{1^2}}{r^2} \right]^{-\frac{1}{2}}$$

$$(r - r^1)^{-1} = \frac{1}{r} \left[1 - \frac{2r^1 \cos\theta}{r} - \frac{r^{1^2}}{r^2} \right]^{-\frac{1}{2}}$$

$$\frac{1}{R} = (r - r^1)^{-1} = \frac{1}{r} \left[1 - \frac{2r^1 \cos\theta}{r} - \frac{r^{1^2}}{r^2} \right]^{-\frac{1}{2}}$$

$$\frac{1}{R} = \frac{1}{r} \left[1 - 2Z\cos\theta - Z^2 \right]^{-\frac{1}{2}} \text{ with } Z = \frac{r^1}{r}$$

As we know that $\left[1 - 2Z\cos\theta - Z^2 \right]^{-\frac{1}{2}}$ is the generating function of Legendre polynomial so that

$$\left[1 - 2Z\cos\theta - Z^2 \right]^{-\frac{1}{2}} = \sum_{n=0}^{\alpha} P_n \cos\theta \; Z^n$$

Therefore

$$\frac{1}{R} = (r - r^1)^{-1} = \frac{1}{r} \sum_{n=0}^{\alpha} P_n \cos\theta \; Z^n$$

$$\frac{1}{R} = (r - r^1)^{-1} = \frac{1}{r} \sum_{n=0}^{\alpha} \left(\frac{r^1}{r} \right)^n P_n \cos\theta \qquad \text{-- 2.50}$$

From equation 2.49, we have

$$A = \frac{\mu_0}{4\pi} I \int \frac{1}{R} dl$$

$$A = \frac{\mu_0}{4\pi} I \int \left(\frac{1}{r} \right) \sum_{n=0}^{\alpha} \left(\frac{r^1}{r} \right)^n P_n \cos\theta \; dl$$

$$A = \frac{\mu_0}{4\pi} I \sum_{n=0}^{\alpha} \left(\frac{1}{r} \right) \int \left(\frac{r^1}{r} \right)^n P_n \cos\theta dl$$

$$A = \frac{\mu_0}{4\pi} I \sum_{n=0}^{\alpha} \left(\frac{1}{r} \right) \frac{1}{r^n} \int (r^1)^n P_n \cos\theta dl$$

$$A = \frac{\mu_0}{4\pi} I \sum_{n=0}^{\alpha} \frac{1}{r^{n+1}} \int (r^1)^n P_n \cos\theta dl$$

$$A = \frac{\mu_0}{4\pi} I \left[\frac{1}{r} \int P_0 \cos\theta dl + \frac{1}{r^2} \int (r^1) P_1 \cos\theta dl + \frac{1}{r^3} \int (r^1)^2 P_2 \cos\theta dl + ... \right]$$

But the Legendre function

$P_0 \cos \theta = 1$

$P_1 \cos \theta = \cos \theta$

$P_2 \cos \theta = \frac{1}{2}(3 \cos^2 \theta - 1)$ and so on.

Thus we get

$$A = \frac{\mu_0}{4\pi} I \left[\frac{1}{r} \int (1) dl + \frac{1}{r^2} \int r^1 \cos\theta dl + \frac{1}{r^3} \int (r^1)^2 \frac{1}{2}(3 \cos^2 \theta - 1) dl + ... \right] \text{---2.51}$$

In equation 2.51, the first term is the monopole term, the second term is the dipole term the third term is the quadrupole term and so on. But the magnetic monopole term is zero. i.e. $\int dl = 0$. This results that there is no magnetic monopole term in nature i.e. $\nabla \cdot B = 0$. In the absence of monopole contribution, the dominant term is the dipole alone. Hence we have

$$A_{Dipole}(r) = \frac{\mu_0}{4\pi} I \left[\frac{1}{r}(0) + \frac{1}{r^2} \int r^1 \cos\theta dl \right]$$

$$A_{Dipole}(r) = \frac{\mu_0 I}{4\pi r^2} \left[\int r^1 \cos\theta dl \right]$$

2.9 Magnetic force due to Volume distribution of current

Line current:

Generally the current in a wire is the amount of charge passing a given point per unit time. Almost in all phenomena involving moving charges the current is defined as the product of charge times velocity. The current is measured in Coulomb per second or Amperes.

$$1 \text{Ampere} = \frac{1 \text{ Coulomb}}{1 \text{ second}}$$

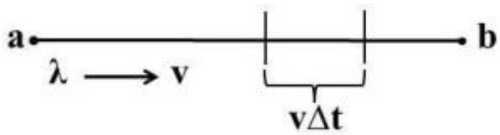


Figure 2.9a

Consider a line charge λ moving through a wire with a velocity v as shown in Figure 2.9a, which constitutes a current I.

$$I = \lambda v \qquad \text{--- 2.52}$$

In Figure 2.8a, a segment of length vΔt, carrying charge λvΔt, passes through any point in a time interval Δt results current I

$$\left[\lambda = \frac{Q}{l} \text{ or } Q = \lambda l, \text{ but } dl = v\Delta t, \ \therefore Q = \lambda v \Delta t\right] \qquad \text{--- 2.53}$$

where λ is the charge density which refers only to the moving charges. The magnetic force on a segment of current-carrying wire is obtained by using Lorentz force law. i.e.

$$F_{Magnetic} = \int (\lambda dl)\,(v \times B) \qquad \text{--- 2.54}$$

$$F_{Magnetic} = (I \times B)\, dl \qquad \text{--- 2.55}$$

Since I, v and dl point in the same direction, the equation 2.55 can be written as

$$F_{Magnetic} = \int I\,(dl \times B) \qquad \text{--- 2.56}$$

But along the length of the wire, the current is constant in magnitude. Hence

$$F_{Magnetic} = I \int (dl \times B) \qquad \text{--- 2.57}$$

Surface current:

When charge flows over a surface, it is described by the surface current density K.

Consider the ribbon of infinitesimal width $dl_\perp$ running parallel to the flow shown in Figure 2.8b.

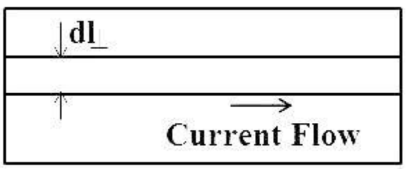


Figure 2.9b

If the current in the ribbon is dI, then the surface current density K is given by

$$K = \frac{dI}{dl_\perp} \qquad \text{--- 2.58}$$

If the surface charge density is σ and its velocity is v, then

$$K = \sigma v \qquad \text{--- 2.59}$$

where K is also called as the current per unit length, perpendicular to flow. Let $\sigma\ dl_\perp$ be the net line charge on the ribbon and hence, $dI = (\sigma\ dl_\perp)\ v$. Thus the magnetic force on a surface current is

$$F_{Magnetic} = \int (\sigma\ da)\ (v \times B)$$

$$F_{Magnetic} = \int (K \times B)\ da \qquad \text{--- 2.60}$$

Volume current:

If the flow of charge is distributed through a dimensional region, then it is described by the volume current density J.

Let us consider a tube of infinitesimal cross section $da_\perp$ moving parallel to the flow as in Figure 2.8c.

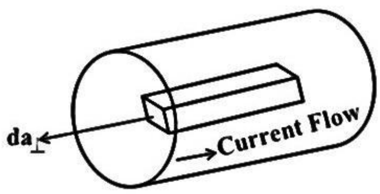


Figure 2.9c

If dI is the current in the tube, then the volume current density is given by

$$J = \frac{dI}{da_\perp} \qquad \text{--- 2.61}$$

where J is the current per unit area-perpendicular to flow.

If the volume charge density is ρ and the velocity is v, then

$$J = \rho v \qquad \text{--- 2.62}$$

Therefore the magnetic force on a volume current is given by

$$F_{Magnetic} = \int (\rho d\tau)\ (v \times B)$$

$$F_{Magnetic} = \int (\rho v \times B)\ d\tau$$

or $$F_{Magnetic} = \int (J \times B)\ d\tau \qquad \text{--- 2.63}$$

2.10 Continuity equation

The current crossing a surface of area is given by

$$I = \int_{Surface} J \cdot da_\perp \ \ (\text{using equation 2.58})$$

or $$I = \int_{Surface} J \cdot da \qquad \text{--- 2.64}$$

The total charge leaving a volume ‘V’ per unit time is

$$\int_{Surface} J \cdot da = \int_{Volume} (\nabla \cdot J)\, d\tau$$

Consider a closed surface ‘S’ enclosing a volume V. If ρ is the volume charge density, then the total charge within the volume is $\int_V \rho \cdot d\tau$. But the transport of charge constitute the current, thus

$$I = \frac{dq}{dt} = \frac{d}{dt} \int_{Volume} \rho \cdot d\tau \qquad \text{--- 2.65}$$

Since the electric charge can neither be created nor destroyed (i.e. the charge is conserved), the flow of the charge out of the volume must be equal to the rate of decrease of the total charge inside the volume. Therefore

$$\int_{Surface} J \cdot da = -\frac{d}{dt} \int_{Volume} \rho \cdot d\tau \qquad \text{--- 2.66}$$

The minus sign indicates that an outward flow decreases the charges left in V. Since ρ is changing with time, we can write

$$\frac{d}{dt} \int_{Volume} \rho \, d\tau = \int_{Volume} \frac{\partial \rho}{\partial t} \, d\tau \qquad \text{--- 2.67}$$

So that the equation 2.66 becomes

$$\int J \cdot da = -\int_{Volume} \left(\frac{\partial \rho}{\partial t} \right) d\tau \qquad \text{--- 2.68}$$

Here the time derivative becomes the partial derivative with respect to time when it is moved inside the integral.

On transforming the surface integral into a volume integral by divergence theorem, we have

$$\int_{Surface} J \cdot da = \int_V (\nabla \cdot J)\, d\tau \qquad \text{--- 2.69}$$

Comparing equations 2.68 and 2.69, we obtain

$$\int_{Volume} (\nabla \cdot J)\, d\tau = -\text{Volume} \int_V \left(\frac{\partial \rho}{\partial t} \right) d\tau \qquad \text{--- 2.70}$$

$$\int_{Volume} (\nabla \cdot J)\, d\tau + \int_{Volume} \left(\frac{\partial \rho}{\partial t}\right) d\tau = 0$$

or $$\int_{Volume} \left[(\nabla \cdot J) + \int_{Volume} \left(\frac{\partial \rho}{\partial t}\right) \right] d\tau = 0 \qquad \text{--- 2.71}$$

This integral must be zero for any orbitrary volume V. Hence the integration must vanish identically. i.e.,

$$(\nabla \cdot J) + \left(\frac{\partial \rho}{\partial t}\right) = 0$$

or $$(\nabla \cdot J) = -\left(\frac{\partial \rho}{\partial t}\right) = 0 \qquad \text{--- 2.72}$$

This is known as continuity equation. It is based on the law of conservation of charge.

In the steady state

$$\left(\frac{\partial \rho}{\partial t}\right) = 0$$

$$\nabla \cdot J = 0$$

This equation is valid only in the region which does not contain a source.

UNIT III : DIELECTRICS

Polarization – The electric field inside a dielectric medium – Gauss law in dielectric and the electric displacement – Electric susceptibility and dielectric constant – Boundary conditions on the field vectors – Dielectric sphere in a uniform electric field- Force on a point charge embedded in a dielectric

3.1 Introduction

In conductors there are large numbers of electrons which are free to move about through the material. Generally many of the electrons are not associated with any particular nucleus but roaming around the material. By contrast, all charges are attached to specific atoms or molecules and they can move only inside the molecule in insulators. Such microscopic displacements are not as intense as the whole arrangement of charge in a conductor. However, the collective effect is responsible for the characteristic behaviour of dielectric materials.

Certain materials exhibit the property that their electrons are not free to move under the influence of an electric field. Such materials are called as insulators. However, the application of the electric field may change the behaviour of an insulator. Thus the insulators whose behaviour gets modified in an electric field are called dielectrics.

When the change in behaviour of the dielectric material is independent of the direction of the applied field, the dielectric is called as isotropic. On the other hand if the change in behaviour of the dielectric depends on the direction of applied field is called as anisotropic.

There are two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule viz. stretching and rotating.

The various properties of dielectric materials are summarised as follows.

(i) When a dielectric is subjected to an external field E, the bound charges shift their relative positions. Due to this displacement, small electric dipoles get induced the dielectric. This is called ‘Polarization’.

(ii) Due to the polarization, the dielectric stores energy.

(iii) Due to the polarization, the flux density of the dielectric increases by an amount equal to the polarization.

(iv) The induced dipoles produce their own electric field and align themselves in the direction of the applied electric field.

(v) The electric field outside and inside the dielectric gets modified due to the induced electric dipoles.

3.2 Polarization

When a piece of dielectric material (polar or nonpolar) is placed in an electric field, tiny dipole moment will be induced by induction in it. Then these elementary dipoles are oriented along the direction of field and the dielectric is said to be polarized.

In the absence of an electric field, the atom is in a position of stable equilibrium. This is because the centre of gravity of its positive charge coincides with that of the negative charge (Figure 3.1a).

Consider a dielectric material in an electric field. The electric field will exert a force on each charged particle. The field will push the positively charged nucleus in the direction of the field and the negatively charged electron cloud in the direction opposite to that of field (Figure 3.1b). The equivalent dipole formed is shown in Figure 3.1c. As a result, the positive and negative parts of each atom or molecule are displaced from their equilibrium position in an opposite direction. One important point to be noted is that induced effect is present only when the electric field is present.

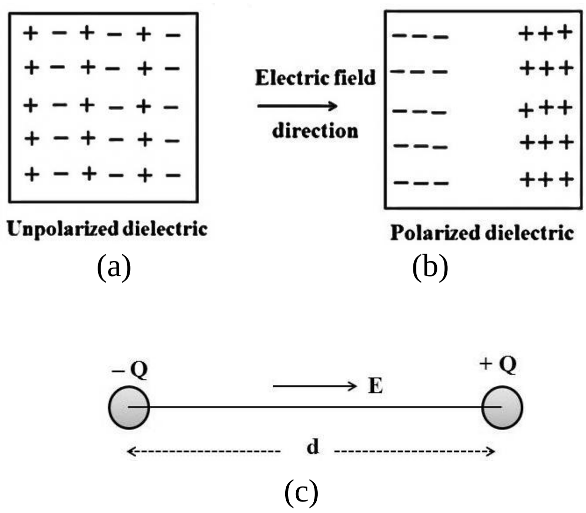


Figure 3.1

In nonpolar molecules, the dipole arrangement is totally absent, in the absence of electric field E. It results only when an externally field E is applied to it. In polar molecules, the permanent displacements between centres of positive and negative charges exist. Thus dipole arrangements exist without application of E. But these dipoles are randomly oriented. Under the application of applied electric field E, the dipoles experience torque and they align with the direction of the applied field E. This is called polarization of polar molecules.

The amount of dipole moment induced will be directly proportional to the field because larger field will displace the charges more than the smaller field. "*The induce dipole moment per unit volume is called the polarization of the medium*" and is represented by P.

A convenient measure of this effect is

$$P = \frac{\text{Dipole moment}}{\text{Unit volume}}$$

Experimentally it has been found that the induced dipole moment P is approximately proportional to the applied electric field E. Therefore

$$P \propto E \qquad \text{---3.1}$$

or

$$P = \alpha E \qquad \text{---3.2}$$

where α is the constant of proportionality and is known as the *"atomic polarizability"*.

3.3 The electric field inside a dielectric medium

In free space, the electric field is defined as the force per unit charge. This implies that the electric field in free space is a measurable quantity. However, to measure the electric field inside a dielectric or other material medium may be very difficult or impractical but, if we confine our attention to the external effects of the dielectric, such internal measurements became unnecessary provided that a theory can be formulated for the behaviour of the dielectric which produces agreement with external measurements. Thus, a distinction should be made between an electrical field as a measurable quantity (as in free space) and an electric filed as a theoretical quantity (as in dielectric).

3.4 Gauss law in dielectric medium and the electric displacement

The total charge density within the dielectric can be written as

$$\rho = \rho_b + \rho_f \qquad \text{--- 3.3}$$

From Gauss's law, we have

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

or

$$\varepsilon_0 (\nabla \cdot E) = \rho = \rho_b + \rho_f \qquad \text{--- 3.4}$$

But the effect of polarization is to produce accumulation of bound charge. The bound charge density is

$$\rho_b = -\nabla \cdot P \qquad \text{--- 3.5}$$

Hence, the equation 3.4 becomes

$$\varepsilon_0(\nabla \cdot E) = -\nabla \cdot P + \rho_f$$

$$\varepsilon_0(\nabla \cdot E) + \nabla \cdot P = \rho_f$$

$$\nabla \cdot (\varepsilon_0 E + P) = \rho_f \quad \text{--- 3.6}$$

The term within the parenthesis is designated by D. Thus the equation 3.6 becomes

$$\nabla \cdot D = \rho_f \quad \text{--- 3.7}$$

Where D = $(\varepsilon_0 E + P)$ is known as the electric displacement.

The equation 3.7 gives Gauss law in terms of D. In an integral form

$$\int_{\text{Surface}} D \cdot da = Q_{f_{\text{Enclosed}}}$$

where $Q_{f_{\text{Enclosed}}}$ represents the total free charges enclosed within the volume.

3.5 Electric susceptibility and dielectric constant

The polarization of a dielectric results from an electric field, which aligns the atomic or molecular dipoles. In many substances the polarization is proportional to the filed E. Thus

$$P \propto E$$

or

$$P = \varepsilon_0 \chi_e E \quad \text{--- 3.8}$$

The equation 3.8 is valid only if the field is not too strong. The constant of proportionality $\chi_e$is called the "electric susceptibility" of the medium. The material which obeys the equation 3.8 is called linear dielectrics.

In linear media, the electric displacement D is given by

$$D = (\varepsilon_0 E + P) \quad \text{--- 3.9}$$

Substituting equation 3.8 in 3.9, we have

$$D = (\varepsilon_0 E + \varepsilon_0 \chi_e E)$$

$$D = \varepsilon_0 E \ (1 + \chi_e)$$

Thus D is also proportional to E. Therefore

$$D = \varepsilon E$$

where $\varepsilon = \varepsilon_0(1 + \chi_e)$is called the permittivity of the material of the dielectric.

In vacuum, there is no matter to polarize. Hence the susceptibility is zero, but the permittivity is ε_0 and is known as the permittivity of free space.

Therefore the permittivity of a dielectric relative to free space is given by

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = (1+\chi_e) \quad \text{--- 3.10}$$

Where ε_r is called the relative permittivity or dielectric constant or specific inductive capacity and is designated by K. Thus

$$K = (1+\chi_e) \quad \text{--- 3.11}$$

The volume bound charge density is given by (Using equation 3.5)

$$\rho_b = -\nabla \cdot P$$

Substituting equation 3.8, we get

$$\rho_b = -\nabla \cdot (\varepsilon_0 \chi_e E)$$

$$\text{But } D = \varepsilon\ E \text{ and therefore } E = \frac{D}{\varepsilon}$$

$$\rho_b = -\nabla \cdot \left(\varepsilon_0 \chi_e \left[\frac{D}{\varepsilon} \right] \right)$$

$$\rho_b = -\nabla \cdot \left(\varepsilon_0 \frac{\chi_e}{\varepsilon} D \right)$$

$$\rho_b = -\nabla \cdot D \left(\varepsilon_0 \frac{\chi_e}{\varepsilon} \right)$$

But $\nabla \cdot D = \rho_f$, hence $\rho_b = -\rho_f \left(\varepsilon_0 \frac{\chi_e}{\varepsilon} \right)$

$$\rho_b = -\rho_f \left(\frac{\chi_e}{1+\chi_e} \right) \left[\because \frac{\varepsilon}{\varepsilon_0} = (1+\chi_e) \text{ or } \frac{\varepsilon_0}{\varepsilon} = \frac{1}{(1+\chi_e)} \right] \quad \text{--- 3.12}$$

Therefore the volume bound charge density is proportional to the density of free charge.

3.6 Boundary conditions on the field vectors

When an electric field or magnetic field passes from one medium to another medium, it is important to study the conditions at the boundary between two media. The conditions existing at the boundary of the two media when field passes from medium to another are called boundary conditions.

Now let us consider the boundary between two perfect dielectrics. The permitivities of the two are ε_1 and ε_2 respectively. The interface is shown in Figure 3.3.

The E and D have to be determined by solving each into two components viz. tangential to the boundary and normal to the surface.

Now consider a closed path abcda rectangular in shape as shown in Figure 3.3. Δh and Δw be the height and width of the shape respectively. It is placed in such a way that the half of the shape i.e. Δh/2 in dielectric medium 1 while the remaining is in dielectric medium 2. Let us calculate E·dl along the path.

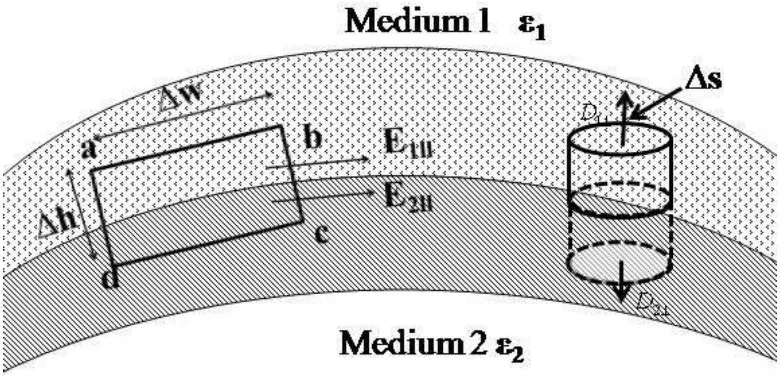


Figure 3.3

Therefore

$$\oint E \cdot dl = 0 \qquad \text{---3.13}$$

Hence $$\int_a^b E \cdot dl + \int_b^c E \cdot dl + \int_c^d E \cdot dl + \int_d^a E \cdot dl = 0 \qquad \text{---3.14}$$

Now $$E_1 = E_{1_{ll}} + E_{1_{\perp}}$$

and $$E_2 = E_{2_{ll}} + E_{2_{\perp}}$$

The electric fields E_1 and E_2have both the parallel and normal components in the respective dielectrics.

When $\Delta h \to 0, \int_b^c$ and $\int_d^a$ become zero as they are line integrals along Δh. Hence equation 3.14 reduces to

$$\int_a^b E \cdot dl + \int_c^d E \cdot dl = 0 \qquad \text{--- 3.15}$$

Now a-b is in dielectric medium 1 hence the corresponding component of E is E_{1ll} as a-b is parallel to the surface. Therefore

$$\int_a^b E \cdot dl = E_{1_{ll}} \int_a^b dl = E_{1_{ll}} \Delta w \qquad \text{--- 3.16}$$

While c-d is in dielectric medium 2 and hence the corresponding component of E is $E_{2_{ll}}$ as c-d direction is also parallel to the surface. But the direction of c-d is opposite to a-b hence corresponding integral is negative of the integral obtained for path a-b

$$\int_c^d E \cdot dl = -E_{2_{ll}} \int_c^d dl = -E_{2_{ll}} \ \Delta w \qquad \text{--- 3.17}$$

Substituting equations 3.16 and 3.17 in 3.15, we obtain

$$E_{1_{ll}} \ \Delta w - E_{2_{ll}} \ \Delta w = 0$$

or $$E_{1_{ll}} \ \Delta w = E_{2_{ll}} \ \Delta w \qquad \text{--- 3.18}$$

Thus the parallel components of electric field intensity at the boundary in both the dielectric remains same. i.e. Electric field intensity is continuous across the boundary.

The relation between the electric field and the electric displacement is D = εE. Hence if D_{1ll} and D_{2ll} are the magnitudes of the parallel components of D in dielectric medium 1 and 2 respectively then,

$$D_{1_{ll}} = \varepsilon_1 E_{1_{ll}} \quad \text{and} \quad D_{2_{ll}} = \varepsilon_2 E_{2_{ll}} \qquad \text{--- 3.19}$$

or $$\frac{D_{1_{ll}}}{D_{2_{ll}}} = \frac{\varepsilon_1}{\varepsilon_2} \quad \left[\because E_{1_{ll}} = E_{2_{ll}}\right] \qquad \text{--- 3.20}$$

Thus the parallel components of D undergo some changes across the interface hence parallel D is said to be discontinuous across the boundary.

Gauss's theorem can be used to determine the normal components. For this consider a Gaussian surface in the form of circular cylinder, placed in such a way that the half of it lies in dielectric medium 1 while the remaining half in dielectric medium 2 as shown in Figure 3.3. When the height Δh→0 the flux leaving from its lateral surface becomes zero. The surface area of its top and bottom of the cylinder is Δs.

$$\int D \cdot ds = Q$$

$$\left[\int_{Top} + \int_{Bottom} + \int_{Latertal}\right] D \cdot ds = Q \qquad \text{---3.21}$$

But $$\left[\int_{Lateral} D \cdot ds\right] = 0 \quad \text{as} \quad \Delta h \to 0 \qquad \text{--- 3.22}$$

Therefore $$\left[\int_{Top} D\cdot ds + \int_{Bottom} D\cdot ds\right] = Q \quad \text{--- 3.23}$$

The flux leaving out normal to the boundary is normal to the top and bottom surfaces. Hence, $D = D_{1_\perp}$ for dielectric medium 1 and $D_{2_\perp}$ for dielectric medium 2. Since the top and bottom surfaces are elementary, flux density can be assumed constant and can be taken out of integration. Therefore

$$\int_{Top} D\cdot ds = D_{1_\perp}\int_{Top} ds = D_{1_\perp}\Delta s \quad \text{--- 3.24}$$

For top surface, the direction of $D_\perp$ is entering the boundary while for bottom surface, the direction of $D_\perp$ is leaving out the boundary. Both the components are in an opposite direction at the boundary.

$$\int_{Bottom} D\cdot ds = -D_{2_\perp}\int_{Bottom} ds = -D_{2_\perp}\Delta s \quad \text{--- 3.25}$$

Using equations 3.24 and 3.25, 3.23 can be written as

$$D_{1_\perp}\Delta s - D_{2_\perp}\Delta s = Q \quad \text{--- 3.26}$$

But the surface charge density $\sigma = \frac{Q}{\Delta s}$ or $Q = \sigma\ \Delta s$.

Therefore $$D_{1_\perp}\Delta s - D_{2_\perp}\Delta s = \sigma \quad \text{--- 3.27}$$

Since no free charge is available in perfect dielectric medium, the available charge on the surface is zero. It means all charges in dielectric are bound charges and are not free. Hence at the dielectric media boundary the surface charge density σ can be assumed zero. i.e. σ = 0. Therefore

$$D_{1_\perp}\Delta s - D_{2_\perp}\Delta s = 0 \quad \text{---3.28}$$

or $$D_{1_\perp}\Delta s = D_{2_\perp}\Delta s$$

$$D_{1_\perp} = D_{2_\perp}$$

Hence, the normal component of flux density D is continuous at the boundary between two perfect dielectric media.

$$D_{1_\perp} = \varepsilon_1 E_{1_\perp} \text{ and } D_{2_\perp} = \varepsilon_2 E_{2_\perp}$$

$$\frac{D_{1_\perp}}{D_{2_\perp}} = \frac{\varepsilon_1 E_{1_\perp}}{\varepsilon_2 E_{2_\perp}} = 1 \quad \text{---3.29}$$

$$\frac{E_{1_\perp}}{E_{2_\perp}} = \frac{\varepsilon_2}{\varepsilon_1} \qquad ---3.30$$

Thus the normal components of the electric field intensity E are inversely proportional to the relative permitivities of the two dielectric media.

3.7 Dielectric sphere in a uniform electric field

Consider a metal sphere of radius a carrying charge Q as shown in Figure 3.4. It is surrounded by a linear dielectric sphere of radius b. The permittivity of the dielectric is ε. The potential at the centre can be computed as follows.

To calculate V, we should know E. The electric field E at any point P which is at a distance r from the charged dielectric is given by

$$E = \frac{Q}{4\pi\varepsilon r^2}\hat{r} \qquad --- 3.31$$

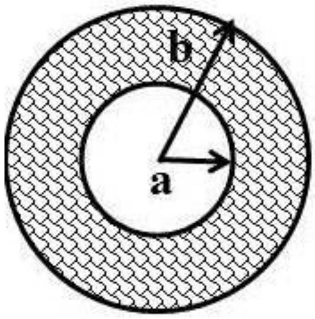


Figure 3.4

But D = εE, hence

$$D = \frac{Q}{4\pi r^2}\hat{r} \text{ for all points } r > a$$

Once r is known, it is very simple to calculate E. Therefore

$$\left.\begin{array}{ll} E = \dfrac{Q}{4\pi\varepsilon r^2}\hat{r} & \text{for } a < r < b \\ E = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & \text{for } r > b \end{array}\right\}$$

Therefore the potential at the centre is

$$V = -\int_\infty^0 E \cdot dl = -\int_\infty^b \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right) dr - \int_b^a \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right) dr - \int_a^0 (0)\, dr$$

$$V = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} + \frac{1}{\varepsilon b}\right) \qquad --- 3.32$$

The polarization P of the dielectric is given by

$$P = \varepsilon_0 \chi_e E = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\hat{r}$$

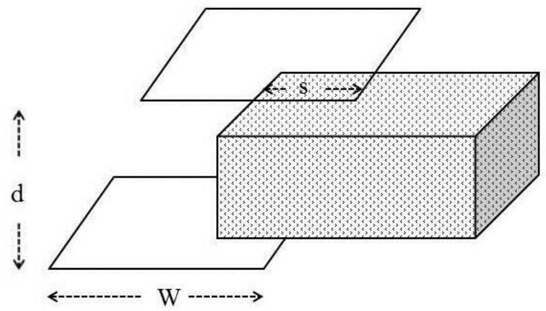
While

$$\sigma_b = P \cdot \hat{n} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon b^2} \quad \text{at the outer surface}$$

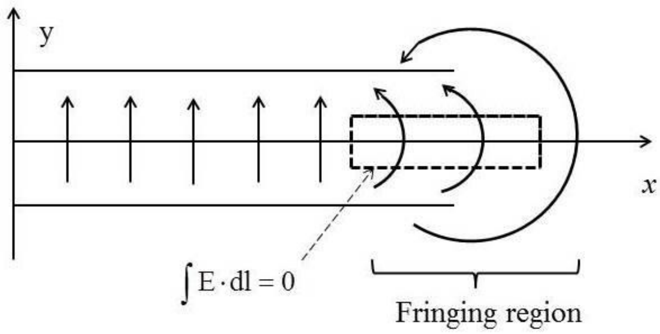
$$\sigma_b = P \cdot \hat{n} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon_0 a^2} \quad \text{at the inner surface}$$

3.8 Force on a point charge embedded in a dielectric

Consider a slab of dielectric material, partially inserted between the plates of a parallel plate capacitor as shown in Figure 3.5a. The field is uniform inside a parallel plat capacitor and zero outside. However, there is a fringing field around the edges (Figure 3.5 b) . It is not easy to determine this fringing filed and this difficulty can be avoided by adopting the following method.



(a)



(b)
Figure 3.5

Let W be the energy of the system. It purely depends upon the overlap distance s. If we pull the dielectric out an infinitesimal distance s towards right, the energy is changed by an amount equal to the amount of work done.

$$dW = F_{us}\, ds \qquad \text{--- 3.34}$$

where F_{us} is the force we exert on dielectrics.

We pull out hardly to overcome the electrical force F on the dielectric.

$$F_{us} = -F$$

Thus the electrical force on the dielectric slab is

$$F = -\frac{dW}{ds} \qquad \text{--- 3.35}$$

The energy stored in the capacitor is

$$W = \frac{1}{2} CV^2 \qquad \text{--- 3.36}$$

In this case the capacitance is given by

$$C = \frac{\varepsilon_0 a}{d}\left(w + \chi_e s\right) \qquad \text{--- 3.37}$$

where w is the length of the plates. As the dielectric moves, the potential will change and the total charge on the plate is constant. i.e. Q = CV. By using equation 3.36 the energy stored in the capacitor in terms of Q is given by

$$W = \frac{1}{2}\frac{C^2V^2}{C} = \frac{1}{2}\frac{Q^2}{C}\left[\because \frac{Q}{C} = V\right] \qquad \text{--- 3.38}$$

Hence, the force F is given by

$$F = -\frac{dW}{ds} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{ds}$$

$$F = \frac{1}{2} V^2 \frac{dC}{ds} \qquad \text{--- 3.39}$$

From equation 3.37, we get

$$\frac{dC}{ds} = \frac{\varepsilon_0 \chi_e a}{d} \qquad \text{--- 3.40}$$

Hence the equation 3.39 becomes

$$F = \frac{1}{2}\ V^2\ \frac{\varepsilon_0 \chi_e a}{d}$$

$$F = \frac{1}{2}\ V^2\ \varepsilon_0 \chi_e \frac{a}{d} \qquad \text{--- 3.41}$$

In computing the force, it is common error to use equation 3.36 with potential V as constant rather than equation 3.38 with charge Q as constant. So one can obtain in general

$$F = -\frac{1}{2}\ V^2\ \varepsilon_0 \chi_e \frac{a}{d} \qquad \text{--- 3.42}$$

and this is off by sign. Then it is also possible to maintain the capacitor at a fixed potential by connecting it with a battery. In that case the battery also does the work as the dielectric moves. Thus instead of equation 3.34, we can have

$$dW = F_{us}\ ds + V\ dQ \qquad \text{--- 3.43}$$

where V dQ is the total work done by the battery. Then it follows that

$$F = -\frac{dW}{ds} + V\frac{dQ}{ds}$$

$$F = -\frac{1}{2}\ V^2\ \frac{dC}{ds} + V\frac{dCV}{ds}$$

$$F = -\frac{1}{2}\ V^2\ \frac{dC}{ds} + V^2\ \frac{dC}{ds}$$

$$F = \frac{1}{2}\ V^2\ \frac{dC}{ds} \qquad \text{--- 3.44}$$

This is the same as equation 3.39 with the correct sign. Therefore we conclude that the force on the dielectric is not dependent whether we keep Q or V as constant. It is purely determined by the distribution of charge. Thus it is very easy to calculate the force F by assuming the charge Q as constant and also not bother about whether work is done by the battery or not.

UNIT IV : MAXWELL'S EQUATION AND PROPAGATION OF EM WAVES

Maxwell's equations and their physical significance–Plane wave equation in homogeneous medium and in free space – Relation between E and H vectors in a uniform plane wave–The wave equation for a conducting medium –Skin depth – Wave propagation in dielectric– Poynting vector – Poynting's theorem.

4.1 Maxwell's equations and their physical significance

James Clerk Maxwell introduced a concept that the magnetic field is produced by a changing electric field. The time varying fields are involved in the experiments of Faraday and the theoretical analysis was done by Maxwell. Maxwell derived four equations to describe the electromagnetic field. These four equations are known as 'Maxwell's equations'. These equations are derived from Gauss's law, Faraday's law, Ampere's circuital law for electric field and Gauss's law for magnetic field. Each of the differential equations has its integral form as counterpart. One form of equation may be derived from the other by using either *'Stokes theorem'* or *'Divergence theorem'*. Maxwell's equations can be derived as follows.

Maxwell's equation - I

Ampere's circuital law:

Ampere's law states that the line integral of magnetic field intensity around any closed path is equal to the current enclosed by the path.

$$\int_{Line} H \cdot dl = I_{Enclosed} \qquad \text{--- 4.1}$$

Replacing current by the surface integral of conduction current density J over an area bounded by the path of integration of H, we get

$$\int_{Line} H \cdot dl = \int_{Surface} J \cdot ds \qquad \text{--- 4.2}$$

In general the total current density involves both conduction current density (J_c) and displacement current density (J_d).

$$J = J_{Conduction}(J_C) + J_{Displacement}\ (J_D) \qquad \text{--- 4.3}$$

Conduction current density J_c:

According to Ohm's law, the conduction current through a resistor R is given by

$$I_C = \frac{V}{R}, \qquad \text{but} \qquad R = \frac{\rho l}{A}$$

where l is the length, A is the area of cross section and ρ is the resistivity of the conductor.

The conductivity of the material of the conductor is given by $\sigma = \frac{1}{\rho}$. Therefore

$$R = \frac{l}{\sigma A}$$

and hence the conduction current, $I_C = \frac{V\sigma A}{l}$

If E is the electric field, then the voltage V = El. Hence

$$I_C = \frac{El\sigma A}{l} = E\sigma A$$

$$\text{or} \quad \frac{I_C}{A} = \sigma E$$

But the conduction current density, $J_C = \frac{I_C}{A}$, Hence

$$J_C = \sigma E \qquad \text{--- 4.4}$$

Displacement current density J_c:

The displacement current through a capacitor is defined as the rate of flow of charge. i.e.

$$I_D = \frac{dQ}{dt} \quad \text{but} \quad Q = CV,$$

Hence $$I_D = \frac{d(CV)}{dt} \quad \text{or} \; I_D = C\frac{dV}{dt}$$

But the capacitance of a parallel plate capacitor is

$$C = \frac{\varepsilon A}{d}$$

where ε is the permittivity of the medium, A is the area of the parallel plate capacitor and d is the distance between the plates. Then the displacement current I_D is

$$I_D = \frac{\varepsilon A}{d}\;\frac{dV}{dt},$$

$$I_D = \frac{\varepsilon A}{d}\;\frac{d(Ed)}{dt}, \; [\because V = Ed]$$

$$\therefore \; I_D = \frac{\varepsilon A}{d}\; d\frac{dE}{dt} \quad \text{or} \quad I_D = \varepsilon A\frac{dE}{dt}$$

$$\frac{I_D}{A} = \varepsilon\frac{dE}{dt}$$

$$\text{But } D = \varepsilon E \quad \text{and} \quad J_D = \frac{I_D}{A} = \varepsilon\frac{dE}{dt} = \frac{d(\varepsilon E)}{dt}$$

Hence

$$J_D = \frac{dD}{dt}[\because D = \varepsilon E] \quad \text{--- 4.5}$$

Substituting equations 4.4 and 4.5 in 4.3, we get

$$J = \sigma E + \frac{dD}{dt} \quad \text{--- 4.6}$$

Now Ampere's law can be written as

$$\int_{Line} H \cdot dl = \int_{Surface} (J_C + J_D) \cdot ds \quad \text{--- 4.7}$$

$$\text{or} \int_{Line} H \cdot dl = \int_{Surface} \left(J_C + \frac{dD}{dt} \right) \cdot ds$$

Unless or otherwise it is not specified by J stands for conduction current density alone, then we can write

$$\int_{Line} H \cdot dl = \int_{Surface} \left(J + \frac{dD}{dt} \right) \cdot ds \quad \text{--- 4.8}$$

$$\int_{Line} H \cdot dl = \int_{Surface} \left(\sigma E + \frac{d(\varepsilon E)}{dt} \right) \cdot ds [\because J = \sigma E \text{ and } D = \varepsilon E]$$

$$\text{or} \int_{Line} H \cdot dl = \int_{Surface} \left(\sigma E + \varepsilon \frac{dE}{dt} \right) \cdot ds \quad \text{--- 4.9}$$

The equation 4.8 is known as integral form of Maxwell's first equation.

*Physical significance:*The magnetomotive force around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.

The differential form of Maxwell's first equation can be obtained by applying Stoke's theorem to equation 4.8.

$$\int_{Line} H \cdot dl = \int_{Surface} (\nabla \times H) \cdot ds \quad \text{--- 4.10}$$

Comparing equations 4.8 and 4.10, we obtain

$$\int_{Surface} (\nabla \times H) \cdot ds = \int_{Surface} \left(J + \frac{dD}{dt} \right) \cdot ds \quad \text{--- 4.11}$$

$$\text{or } (\nabla \times H) = \left(J + \frac{dD}{dt} \right) \quad \text{--- 4.12}$$

The equation 4.11 can also be written as

$$\int_{Surface} (\nabla \times H) \cdot ds = \int_{Surface} \left(\sigma E + \varepsilon \frac{dE}{dt} \right) \cdot ds \quad \text{--- 4.13}$$

or $$(\nabla \times H) = \left(\sigma E + \varepsilon \frac{dE}{dt}\right) \quad \text{--- 4.14}$$

This is known as differential form of Maxwell's first equation.

Maxwell's equation – II

Faraday's law: Faraday's law states that the induced emf in a circuit is equal to the time rate of decrease of total magnetic flux linked through the circuit.

$$e = -\frac{d\phi}{dt}, \quad \text{but } \phi = \int_{Surface} B \cdot ds$$

$$e = -\frac{d}{dt}\left(\int_{Surface} B \cdot ds\right) \quad \text{--- 4.15}$$

We know that emf 'e' induced in a circuit can be represented as the line integral of the electric field intensity around the close path. i.e.

$$e = \int_{Line} E \cdot dl \quad \text{--- 4.16}$$

Comparing equations 4.15 and 4.16, we have

$$\int_{Line} E \cdot dl = -\frac{d}{dt}\left(\int_{Surface} B \cdot ds\right) \quad \text{--- 4.17}$$

$$\int_{Line} E \cdot dl = -\left(\int_{Surface} \frac{\partial B}{\partial t} \cdot ds\right) \quad \text{--- 4.18}$$

Comparing equations 4.15 and 4.18, we obtain

$$\int_{Line} E \cdot dl = -\frac{d\phi}{dt} = e$$

From equation 4.18, we obtain

$$\int_{Line} E \cdot dl = -\mu\left(\int_{Surface} \frac{\partial H}{\partial t} \cdot ds\right) \quad [\because B = \mu H] \quad \text{--- 4.19}$$

This is integral form of Maxwell's second equation.

*Physical significance:*The emf induced around a closed path is equal to the negative rate of change of magnetic flux linked with the path.

By applying Stoke's theorem

$$\int_{Line} E \cdot dl = \int_{Surface} (\nabla \times E) \cdot ds \quad \text{--- 4.20}$$

Comparing equations 4.18 and 4.20, we obtain

$$\int_{Surface} (\nabla \times E) \cdot ds = -\left(\int_{Surface} \frac{\partial B}{\partial t} \cdot ds \right)$$

or $$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{--- 4.21}$$

or $$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad \text{--- 4.22}$$

This is the differential form of Maxwell's second equation.

Maxwell's equation – III

Gauss's law in electrostatics:The total electric flux emerging out of the closed surface is equal to the total charge enclosed by the surface. This can be written in integral form as

$$\int_{Surface} D \cdot ds = Q_{Enclosed} \quad \text{--- 4.23}$$

$Q_{Enclosed}$can be replaced by the volume integral of the charge density ρ. Therefore the charge enclosed by the given closed surface.

$$\text{i.e. } Q_{Enclosed} = \int_{V} \rho \cdot dv \quad \text{--- 4.24}$$

Now, equation 4.23 becomes as

$$\int_{Surface} D \cdot ds = \int_{Volume} \rho \cdot dv \quad \text{--- 4.25}$$

$$\text{or } \int_{Surface} E \cdot ds = \int_{Volume} \frac{\rho}{\varepsilon} \cdot dv \quad [\because D = \varepsilon E]$$

This is integral form of Maxwell's third equation.

Using divergence theorem, we can write

$$\int_{Surface} D \cdot ds = \int_{Volume} (\nabla \cdot D)\, d\tau \quad \text{--- 4.26}$$

Comparing the above equations 4.25 and 4.26, we obtain

$$\int_{Volume} (\nabla \cdot D)\, d\tau = \int_{Volume} \rho \cdot dv \quad \text{---4.27}$$

Assuming same volume for integration on both sides, we have

$$\nabla \cdot D = \rho \quad \text{--- 4.28}$$

This is the differential form of Maxwell's third equation. In other words

$$\nabla \cdot (\varepsilon E) = \rho \; [\because D = \varepsilon\, E]$$

$$\nabla \cdot E = \frac{\rho}{\varepsilon} \quad \text{--- 4.29}$$

*Physical significance:*The outward flux of displacement vector through the surface enclosing a volume is equal to the net charge contained within the volume.

Maxwell's equation – IV

Magnetic Gauss' law: The total magnetic flux around any closed surface is equal to zero.

$$\int_{Surface} B \cdot ds = 0 \qquad \text{--- 4.30}$$

This is Maxwell's equation in integral form.

By applying Divergence theorem, we have

$$\int_{Surface} B \cdot ds = \int_{Volume} (\nabla \cdot B)\, d\tau \qquad \text{--- 4.31}$$

Comparing equations 4.30 and 4.31, we obtain

$$\int_{Volume} (\nabla \cdot B)\, d\tau = 0$$

or

$$(\nabla \cdot B) = 0 \qquad \text{--- 4.32}$$

This is differential form of Maxwell's fourth equation.

*Physical significance:*Net outward flux of magnetic induction vector through the surface enclosing a volume is zero.

Maxwell's equations are summarized as shown in Table 1.

Table 1. Maxwell's equations in a conducting medium

Integral form	Differential form	Significance
$\int_{Line} H \cdot dl = \int_{Surface} \left(J + \frac{dD}{dt} \right) \cdot ds$ or $\int_{Line} H \cdot dl = \int_{Surface} \left(\sigma E + \varepsilon \frac{dE}{dt} \right) \cdot ds$	or $\nabla \times H = J + \frac{dD}{dt}$ or or $\nabla \times H = \sigma E + \varepsilon \frac{dE}{dt}$	Ampere's circuital law
$\int_{Line} E \cdot dl = -\left(\int_{Surface} \frac{\partial B}{\partial t} \cdot ds \right)$ or $\int_{Line} E \cdot dl = -\mu \left(\int_{Surface} \frac{\partial H}{\partial t} \cdot ds \right)$	$\nabla \times E = -\frac{\partial B}{\partial t}$ or $\nabla \times E = -\mu \frac{\partial H}{\partial t}$	Faraday's Law
$\int_{Surface} D \cdot ds = \int_{Volume} \rho \cdot dv$ or $\int_{Surface} E \cdot ds = \int_{Volume} \frac{\rho}{\varepsilon} \cdot dv \quad [\because D = \varepsilon E]$	$\nabla \cdot D = \rho$ $\nabla \cdot E = \frac{\rho}{\varepsilon}$	Gauss's law
$\int_{Volume} (\nabla \cdot B)\, d\tau = 0$	$\nabla \cdot B = 0$	No isolated magnetic charge

4.1.1 Maxwell equations in free space

In a free space, there are no charges enclosed. Free space is a non-conducting medium (i.e. dielectric) in which volume charge density ρ and conductivity σ are zero. Maxwell's equations in free space are summarized as shown in Table 2.

Table 2. Maxwell's equations in free space

Integral form	Differential form
$\int_{Line} H \cdot dl = \int_{Surface} \left(\frac{dD}{dt}\right) \cdot ds$ or $\int_{Line} H \cdot dl = \int_{Surface} \left(\varepsilon \frac{dE}{dt}\right) \cdot ds$	or $\nabla \times H = \frac{dD}{dt}$ or or $\nabla \times H = \varepsilon \frac{dE}{dt}$
$\int_{Line} E \cdot dl = -\left(\int_{Surface} \frac{\partial B}{\partial t} \cdot ds\right)$ or $\int_{Line} E \cdot dl = -\mu \left(\int_{Surface} \frac{\partial H}{\partial t} \cdot ds\right)$	$\nabla \times E = -\frac{\partial B}{\partial t}$ or $\nabla \times E = -\mu \frac{\partial H}{\partial t}$
$\int_{Surface} D \cdot ds = 0$ or $\int_{Surface} E \cdot ds = 0$	$\nabla \cdot D = 0$ or $\nabla \cdot E = 0$
$\int_{Volume} (\nabla \cdot B)\, d\tau = 0$	$\nabla \cdot B = 0$

4.2 Plane wave equation in homogeneous medium and in free space

The propagation of electromagnetic wave can be explained easily with the help of Maxwell's equations. Electromagnetic wave equation can also be obtained from Maxwell's equations.

4.2.1 In homogeneous medium

From differential form of Maxwell's second equation, we have

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

or

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

Taking curl on both sides, we get

$$\nabla \times \nabla \times E = \nabla \times \left(-\mu \frac{\partial H}{\partial t}\right)$$

$$\nabla \times \nabla \times E = -\mu \left[\nabla \times \left(\frac{\partial H}{\partial t}\right)\right] \qquad \text{--- 4.33}$$

From differential form of Maxwell's first equation, we have

or $$\nabla \times H = \sigma E + \varepsilon \frac{dE}{dt}$$

Differentiating

$$\frac{\partial}{\partial t}(\nabla \times H) = \frac{\partial}{\partial t}\left(\sigma E + \varepsilon \frac{\partial E}{\partial t}\right)$$

$$\nabla \times \left(\frac{\partial H}{\partial t}\right) = \sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \text{--- 4.34}$$

Substituting equation 4.34 in 4.33

$$\nabla \times \nabla \times E = -\mu \left[\sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2}\right]$$

$$\nabla \times \nabla \times E = -\mu\, \sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \text{--- 4.35}$$

From vector identity, we have

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E \qquad \text{--- 4.36}$$

Since there is no net charge within the conductor i.e. most of the region are source or charge free then the charge density $\rho = 0$. Therefore $\nabla \cdot E = 0$ and hence $\nabla (\nabla \cdot E) = 0$. Now equation 4.36 is reduced as

$$\nabla \times \nabla \times E = -\nabla^2 E \qquad \text{--- 4.37}$$

Comparing equations 4.35 and 4.37, we obtain

$$-\nabla^2 E = -\mu\, \sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

or $$\nabla^2 E = \mu\, \sigma \frac{\partial E}{\partial t} + \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \text{--- 4.38}$$

or $$\nabla^2 E - \mu\, \sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad \text{--- 4.39}$$

The equation 4.39 is known as the *'wave equation for electric field E'*.

The wave equation for H is obtained in a similar manner as above.

The differential form of Maxwell's first equation is

$$\nabla \times H = \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

Taking curl on both sides, we get

$$\nabla \times \nabla \times H = \nabla \times \left(\sigma E + \varepsilon \frac{\partial E}{\partial t} \right)$$

$$\nabla \times \nabla \times H = \sigma(\nabla \times E) + \varepsilon \left(\nabla \times \frac{\partial E}{\partial t} \right) \quad \text{--- 4.40}$$

From differential form of Maxwell's second equation, we get

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad \text{--- 4.41}$$

Differentiating

$$\nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial^2 H}{\partial t^2} \quad \text{--- 4.42}$$

Substituting equations 4.41 and 4.42 in 4.40, we obtain

$$\nabla \times \nabla \times H = -\sigma\mu \frac{\partial H}{\partial t} - \varepsilon\mu \left(\frac{\partial^2 H}{\partial t^2} \right) \quad \text{--- 4.43}$$

But from the vector identity formula

$$\nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H$$

But from Maxwell's fourth equation

$$\nabla \cdot B = \nabla \cdot (\mu H) = 0$$

or $\nabla \cdot B = \mu\, (\nabla \cdot H) = 0$

Hence $\nabla \times \nabla \times H = -\nabla^2 H$ --- 4.44

Comparing equations 4.43 and 4.44

$$-\nabla^2 H = -\sigma\mu \frac{\partial H}{\partial t} - \varepsilon\mu \left(\frac{\partial^2 H}{\partial t^2} \right) \quad \text{--- 4.45}$$

$$\nabla^2 H = \sigma\mu \frac{\partial H}{\partial t} + \varepsilon\mu \left(\frac{\partial^2 H}{\partial t^2} \right)$$

$$\nabla^2 H - \sigma\mu \frac{\partial H}{\partial t} - \varepsilon\mu \left(\frac{\partial^2 H}{\partial t^2} \right) = 0 \quad \text{--- 4.46}$$

This is the wave equation for the magnetic field intensity H.

4.2.2 In free space

Since there is no charge in free space (i.e. dielectric medium), the charge density (ρ) and hence the current density (J) is zero. The electromagnetic wave equations can be obtained from differential form of Maxwell's equations.

The wave equation for electric field is

$$\nabla^2 E - \mu\,\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

$$\nabla^2 E - \mu\,(0) \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \quad [\sigma = 0] \text{--- 4.47}$$

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad \text{--- 4.48}$$

This is the wave equation for free space in terms of electric field E.

To obtain the wave equation for free space in terms of magnetic field, consider the wave equation for magnetic field intensity

$$\nabla^2 H - \sigma\mu \frac{\partial H}{\partial t} - \varepsilon\mu \left(\frac{\partial^2 H}{\partial t^2}\right) = 0$$

$$\nabla^2 H - (0)\mu \frac{\partial H}{\partial t} - \varepsilon\mu \left(\frac{\partial^2 H}{\partial t^2}\right) = 0 \quad [\sigma = 0] \qquad \text{--- 4.49}$$

$$\nabla^2 H - \varepsilon\mu \left(\frac{\partial^2 H}{\partial t^2}\right) = 0 \qquad \text{--- 4.50}$$

This is the wave equation for free space in terms of magnetic field H.

The value of $\mu_r = 1$ for free space and $\varepsilon_r = 1$ for air, then wave equation 4.48 becomes

$$\nabla^2 E - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0$$

$$\text{But } \mu_0\varepsilon_0 = \left(4\pi \times 10^{-7}\right) \times \left(\frac{1}{36\pi \times 10^{-9}}\right)$$

$$\mu_0\varepsilon_0 = \frac{1}{9 \times 10^{16}} \text{ or } \sqrt{\mu_0\varepsilon_0} = \frac{1}{3 \times 10^8}$$

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8 = c \quad \text{m/sec} \qquad \text{--- 4.51}$$

where c is the velocity of light. Hence the wave equation becomes

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \qquad \text{--- 4.52}$$

and

$$\nabla^2 H - \frac{1}{c^2} \frac{\partial^2 H}{\partial t^2} = 0 \qquad \text{--- 4.53}$$

4.3 Relation between E and H vectors in a uniform plane wave

A linearly polarized, uniform plane wave is one which satisfies the following conditions.

(a) At every point in space, E and H are perpendicular to each other and also to the direction of propagation of energy.

(b) The fields vary harmonically with time and at the same frequency everywhere in space.

(c) Each field has the same direction, magnitude and phase at every point in any plane perpendicular to the direction of wave propagation.

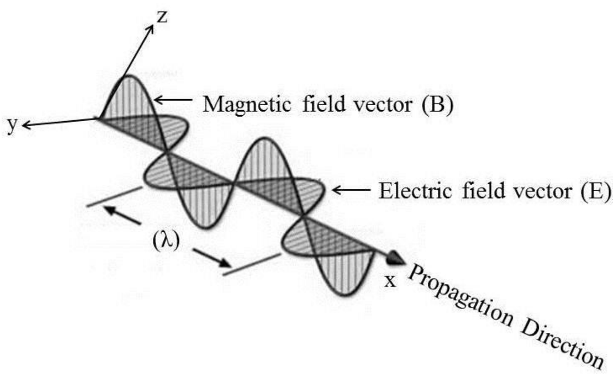


Figure 4.1

Consider a wave travelling along x-direction as shown in Figure 4.1. It is noted that E and H are not having x-component along the direction of propagation of electromagnetic wave. From Faraday's law, we have

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$LHS = \nabla \times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\nabla \times E = \hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{j}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\nabla \times E = \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\left[\begin{array}{l} \because \text{all other values are zero, since E is acting} \\ \text{only in Y direction, } E_x = E_z = 0 \end{array}\right]$$

This must be equal to $-\frac{\partial B_z}{\partial t}$

By right hand screw rule, we know that if E is acting in Y direction and propagating in x direction, then the magnetic field intensity H will be in positive z direction.

Therefore we conclude that

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\frac{\partial B_z}{\partial t} \qquad \text{--- 4.54}$$

where B is acting in positive z direction.

Let E_y be equal to sin (x-vt). Where v is the velocity, t is the time and x is the direction of propagationof electromagnetic wave. Hence we write

$$E_y = \sin(x-vt) \qquad \text{--- 4.55}$$

The above relation means E_y is propagating in the positive y direction

$$\frac{\partial E_y}{\partial x} = \cos(x - vt) \qquad \text{--- 4.56}$$

Using equation 4.54, 4.56 can be taken as

$$\cos(x - vt) = -\frac{\partial B_z}{\partial t}$$

$$B_z = -\int \cos(x - vt)\, dt$$

$$B_z = \frac{\sin\,(x - vt)}{v}$$

Using equation 4.55, we may write

$$B = \left(\frac{1}{v}\right)E \qquad \text{---4.57}$$

This the relation between E and B, But we know, B=μH, hence we have

$$\mu H = \left(\frac{1}{v}\right)E \qquad \text{---4.58}$$

Moreover

$$v = \left(\frac{1}{\sqrt{\mu\varepsilon}}\right)$$

$$\mu H = \sqrt{\mu\varepsilon}\;\; E \qquad \text{--- 4.59}$$

or

$$H = \sqrt{\frac{\varepsilon}{\mu}}\; E$$

or

$$E = \sqrt{\frac{\mu}{\varepsilon}}\; H \qquad \text{--- 4.60}$$

If the wave is propagating in free space, we have

$$E = \sqrt{\frac{\mu_0}{\varepsilon_0}}\; H$$

$$E = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}}\ H$$

$$\frac{E}{H} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}} = 377\ \Omega = Z_0$$

where Z_0 is called the 'wave impedance' of free space.

From equation 4.60, we have

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{--- 4.61}$$

Equation 4.61 states that in a travelling plane wave there is a definite ratio between the amplitude of E and H. This ratio is equal to the square root of the ratio of permeability to the dielectric constant of the medium. $\sqrt{\frac{\mu}{\varepsilon}}$ has the unit of ohm as E is in Volt/m and H is in Amp/m. The ratio $\sqrt{\frac{\mu}{\varepsilon}}$ is known as characteristic impedance, η, or the intrinsic impedance of the medium.

4.4 The wave equation for a conducting medium –Skin depth

In free space, the charge density ρ_f and the free current density J_f are zero. But in the case of conductors, ρ_f and J_f are not zero. According to Ohm's law, the current density in a conductor is proportional to the electric field.

$$J_f = \sigma E \qquad \text{--- 4.62}$$

Therefore the Maxwell's equations for linear media are

$$\left.\begin{aligned} &\text{(i) } \nabla \cdot E = \frac{\rho_f}{\varepsilon} \\ &\text{(ii) } \nabla \cdot B = 0 \\ &\text{(iii) } \nabla \times E = -\frac{\partial B}{\partial t} \\ &\text{(iv) } \nabla \times B = \mu\sigma E + \mu\varepsilon \frac{\partial E}{\partial t} \end{aligned}\right\} \qquad \text{--- 4.63}$$

Now the continuity equation for free current is given by

$$\nabla \cdot J_f = -\frac{\partial \rho_f}{\partial t} \text{ or } \nabla.(\sigma E) = -\frac{\partial \rho_f}{\partial t} \text{ or } \sigma(\nabla \cdot E) = -\frac{\partial \rho_f}{\partial t} \qquad \text{--- 4.64}$$

$$\sigma\left(\frac{\rho_f}{\varepsilon}\right) = -\frac{\partial \rho_f}{\partial t}; \quad \text{or} \quad \left(\frac{\sigma}{\varepsilon}\right)\rho_f = -\frac{\partial \rho_f}{\partial t}$$

$$\frac{\partial \rho_f}{\partial t} = -\left(\frac{\sigma}{\varepsilon}\right)\rho_f \quad \text{or} \quad \frac{\partial \rho_f}{\rho_f} = -\left(\frac{\sigma}{\varepsilon}\right)\partial t \qquad \text{--- 4.65}$$

Now integrating equation 4.65, we get

$$\log \rho_f = -\left(\frac{\sigma}{\varepsilon}\right)t + C$$

When t=0, log ρ_f= log $\rho(0)$. Hence $\log \rho_f(0) = -\frac{\sigma}{\varepsilon}\cdot 0 + C$

$$\therefore C = \log \rho_f(0)$$

Thus $$\log \rho_f = -\left(\frac{\sigma}{\varepsilon}\right)t + \log \rho_f(0)$$

$$\log \rho_f - \log \rho_f(0) = -\left(\frac{\sigma}{\varepsilon}\right)t$$

$$\text{or} \quad \log \frac{\rho_f}{\rho_{f(0)}} = -\left(\frac{\sigma}{\varepsilon}\right)t$$

$$\frac{\rho_f}{\rho_{f(0)}} = e^{-\left(\frac{\sigma}{\varepsilon}\right)t}$$

Therefore $$\rho_f = e^{-\left(\frac{\sigma}{\varepsilon}\right)t}\cdot \rho_{f(0)}$$

$$\rho_f = \rho_{f(0)}\, e^{-\left(\frac{\sigma}{\varepsilon}\right)t} \qquad \text{--- 4.66}$$

The initial free charge density $\rho_{f(0)}$ dissipates in a characteristic time $\tau = (\varepsilon/\sigma)$. Put ρ_f= 0, then Maxwell's equations become

(i) $\nabla \cdot E = 0$

(ii) $\nabla \cdot B = 0$

(iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

(iv) $\nabla \times B = \mu\sigma E + \mu\varepsilon\frac{\partial E}{\partial t}$

--- 4.67

Applying curl to equation 4.67 (iii) and (iv), we obtain

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\frac{\partial B}{\partial t}\right)$$

$$\nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t}(\nabla \times B)$$

$$-\nabla^2 E = -\frac{\partial}{\partial t}\left(\mu\sigma E + \mu\varepsilon\frac{\partial E}{\partial t}\right)$$

$$-\nabla^2 E = -\mu\sigma\frac{\partial E}{\partial t} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E = \mu\sigma\frac{\partial E}{\partial t} + \mu\varepsilon\frac{\partial^2 E}{\partial t^2} \qquad \text{--- 4.68}$$

Similarly we have

$$\nabla^2 B = \mu\sigma\frac{\partial B}{\partial t} + \mu\varepsilon\frac{\partial^2 B}{\partial t^2} \qquad \text{--- 4.69}$$

The solutions for the electric and magnetic fields of a plane wave are given by

$$E\ (x,t) = E_0 e^{i(kx-wt)} \text{ and } B\ (x,t) = B_0 e^{i(kx-wt)}$$

Substituting these values into the wave equations 4.68, we have

$$\nabla^2\left(E_0 e^{i(kx-wt)}\right) = \mu\sigma\frac{\partial}{\partial t}\left(E_0 e^{i(kx-wt)}\right) + \mu\varepsilon\frac{\partial^2}{\partial t^2}\left(E_0 e^{i(kx-wt)}\right)$$

$$E_0\ i^2\, k^2 e^{i(kx-wt)} = \mu\sigma E_0(-i\omega)\ e^{i(kx-wt)} + \mu\varepsilon E_0\, i^2\, \omega^2\, e^{i(kx-wt)}$$

$$(-1)k^2 E_0\ e^{i(kx-wt)} = -i\mu\sigma\omega\, E_0\ e^{i(kx-wt)} - \mu\varepsilon\ \omega^2\ E_0 e^{i(kx-wt)}$$

Cancelling the common term $E_0\ e^{i(kx-wt)}$, we get

$$-k^2 \quad = \quad -i\mu\sigma\omega \quad -\mu\varepsilon\,\omega^2$$

or

$$k^2 = i\mu\sigma\omega + \mu\varepsilon\omega^2 \qquad \text{---4.70}$$

Let $k = k_+ + ik_-$, such that $k^2 = (k_+ + ik_-)^2$

$$k^2 = \left(k_+{}^2 + i^2 k_-{}^2 + 2k_+ ik_-\right) = \left(k_+{}^2 - k_-{}^2 + 2k_+ ik_-\right)$$

$$k^2 = \left(k_+{}^2 - k_-{}^2\right) + i(2k_+ k_-) \qquad \text{--- 4.71}$$

Comparing the real and imaginary parts of equations 4.70 and 4.71, we have

$$k_+{}^2 - k_-{}^2 = \mu\varepsilon\omega^2 \quad \text{and} \quad (2k_+ k_-) = \mu\sigma\omega$$

Hence

$$k_+{}^2 = \mu\varepsilon\omega^2 + k_-{}^2 \text{ and } (2k_+\ k_-)^2 = (\mu\sigma\omega)^2$$

$$4k_+{}^2\ k_-{}^2 = (\mu\sigma\omega)^2$$

$$4\left(\mu\varepsilon\omega^2 + k_-{}^2\right) k_-{}^2 = (\mu\sigma\omega)^2$$

$$\left(4\mu\varepsilon\omega^2 k_-{}^2 + 4k_-{}^4\right) = (\mu\sigma\omega)^2$$

$$4\mu\varepsilon\omega^2 k_-{}^2 + 4k_-{}^4 - (\mu\sigma\omega)^2 = 0$$

$$4k_-{}^4 + 4\mu\varepsilon\omega^2 k_-{}^2 - (\mu\sigma\omega)^2 = 0$$

$$k_-^{\,2} = \frac{-4\mu\varepsilon\omega^2 \pm \sqrt{\left(4\mu\varepsilon\omega^2\right)^2 - 4\cdot 4\left(\mu\sigma\omega\right)^2}}{2\cdot 4}$$

$$k_-^{\,2} = \frac{-4\mu\varepsilon\omega^2}{8} \pm \frac{\sqrt{16\left(\mu\varepsilon\omega^2\right)^2 - 16\left(\mu\sigma\omega\right)^2}}{8}$$

$$k_-^{\,2} = \frac{-\mu\varepsilon\omega^2}{2} \pm \frac{\sqrt{16}}{8}\sqrt{\left(\mu\varepsilon\omega^2\right)^2 - \left(\mu\sigma\omega\right)^2}$$

$$k_-^{\,2} = \frac{-\mu\varepsilon\omega^2}{2} \pm \frac{4}{8}\sqrt{\left(\mu\varepsilon\omega^2\right)^2 - \left(\mu\sigma\omega\right)^2}$$

$$k_-^{\,2} = \frac{-\mu\varepsilon\omega^2}{2} \pm \frac{1}{2}\sqrt{\mu^2\varepsilon^2\omega^2\omega^2 + \left(\mu\sigma\omega\right)^2}$$

$$k_-^{\,2} = \frac{-\mu\varepsilon\omega^2}{2} \pm \frac{1}{2}\sqrt{\mu^2\varepsilon^2\omega^2\omega^2 + \mu^2\sigma^2\omega^2}$$

$$k_-^{\,2} = \frac{-\mu\varepsilon\omega^2}{2} \pm \frac{1}{2}\sqrt{\mu^2\varepsilon^2\omega^2\omega^2\left(1 + \frac{\sigma^2}{\varepsilon^2\omega^2}\right)}$$

$$k_-^{\,2} = \frac{-\mu\varepsilon\omega^2}{2} \pm \frac{1}{2}\sqrt{\left(\mu\varepsilon\omega^2\right)^2\left(1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right)}$$

$$k_-^{\,2} = \frac{-\mu\varepsilon\omega^2}{2} \pm \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\sqrt{\left(1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right)}$$

$$k_-^{\,2} = -\frac{1}{2}\left(\mu\varepsilon\omega^2\right) \pm \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}$$

$$k_-^{\,2} = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[-1 \pm \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]$$

$$k_-^{\,2} = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]$$

$$k_- = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{\frac{1}{2}} \qquad \text{--- 4.72}$$

Similarly we get,

$$k_+ = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{\frac{1}{2}} \qquad \text{--- 4.73}$$

The imaginary part of k (i.e. k_-) results in an attenuation of the wave i.e. decreasing amplitude with increasing x. Then

$$E\ (x,t) = E_0 e^{-k_- x}\, e^{i(kx-wt)} \text{ and } B\ (x,t) = B_0 e^{-k_- x}\, e^{i(kx-wt)} \qquad \text{--- 4.74}$$

Greater is the value of k_-,greater is the attenuation. The term $\frac{1}{k_-}$ measures the depth at which the electromagnetic wave entering a conductor is damped to $\frac{1}{e} = \frac{1}{2.718} = 0.369$ of its initial at the surface. This depth is known as the *"skin depth"* or the penetration depth. It is usually represented by 'd'. Therefore

$$d = \left(\frac{1}{k_-}\right) \qquad \text{--- 4.75}$$

$$d = \frac{1}{\omega\sqrt{\frac{\mu\varepsilon}{2}}} \times \frac{1}{\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{\frac{1}{2}}}$$

$$d = \frac{1}{\omega\sqrt{\frac{\mu\varepsilon}{2}}} \times \frac{1}{\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \quad - 1\right]^{\frac{1}{2}}}$$

--- 4.76

$$d = \frac{1}{\omega\sqrt{\frac{\mu\varepsilon}{2}}} \times \left[\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right) \Big/ 2\right]^{-\frac{1}{2}} \qquad \text{--- 4.77}$$

Thus 'd' measures the depth at which an electromagnetic wave penetrates into the conductor.

The real part of k i.e. k_+determines the wavelength, the propagation speed and the index of refraction. Therefore

$$\lambda = \frac{2\pi}{k_+}, \qquad v = \frac{\omega}{k_+} \quad \text{and} \quad n = \frac{ck_+}{\omega} \qquad \text{--- 4.78}$$

For a good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1$, therefore $\sigma >> \omega\varepsilon$. Hence

$$k_+ \cong k_- \cong \sqrt{\frac{\omega\sigma\mu}{2}} \qquad \text{--- 4.79}$$

The skin depth decreases with increase in frequency.

The good and poor conductors depend on the frequency i.e. the substance can be good conductor at low frequency and a poor one at high frequency.

4.5 Wave propagation in dielectrics

For a poor conductor, i.e. good dielectrics, $\frac{\sigma}{\omega e} << 1$, therefore σ <<ωε. . Hence

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} = \sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} = \left[1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right]^{\frac{1}{2}} \approx \left[1+\frac{\sigma^2}{2\omega^2\varepsilon^2}\right]$$

But from equation 4.73

$$k_+ = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{\frac{1}{2}} = \omega\left[\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right)\right]^{\frac{1}{2}}$$

$$k_+ = \omega\left[\frac{\mu\varepsilon}{2}\left(\left(1+\frac{\sigma^2}{2\varepsilon^2\omega^2}\right)+1\right)\right]^{\frac{1}{2}}$$

$$k_+ = \omega\left[\frac{\mu\varepsilon}{2}\left(2+\frac{\sigma^2}{2\varepsilon^2\omega^2}\right)\right]^{\frac{1}{2}}$$

$$k_+ = \omega\left[\mu\varepsilon\left(1+\frac{\sigma^2}{4\varepsilon^2\omega^2}\right)\right]^{\frac{1}{2}}$$

$$k_+ = \omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma^2}{4\varepsilon^2\omega^2}\right]^{\frac{1}{2}}$$

$$k_+ = \omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma^2}{8\varepsilon^2\omega^2}\right]$$

The velocity of the electromagnetic wave in dielectric medium is

$$v = \frac{\omega}{k_+} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma^2}{8\varepsilon^2\omega^2}\right]}$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}}\left[1-\frac{\sigma^2}{8\varepsilon^2\omega^2}\right]$$

$$v = v_0\left[1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right] \quad \left[\because v_0 = \frac{1}{\sqrt{\mu\varepsilon}}\right]$$

Then

$$k_{-} = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^{2}}-1\right]^{\frac{1}{2}}$$

$$k_{-} = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^{2}}-1\right]^{\frac{1}{2}} = \omega\left[\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^{2}}-1\right)\right]^{\frac{1}{2}}$$

$$k_{+} = \omega\left[\frac{\mu\varepsilon}{2}\left(\left(1+\frac{\sigma^{2}}{2\varepsilon^{2}\omega^{2}}-1\right)\right)\right]^{\frac{1}{2}}$$

$$k_{+} = \omega\left[\frac{\mu\varepsilon}{2}\left(\frac{\sigma^{2}}{2\varepsilon^{2}\omega^{2}}\right)\right]^{\frac{1}{2}}$$

$$k_{+} = \omega\left[\sqrt{\frac{\mu\varepsilon}{2}}\left(\frac{\sigma^{2}}{2\varepsilon^{2}\omega^{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$k_{+} = \omega\left[\frac{\mu\varepsilon\sigma^{2}}{4\varepsilon^{2}\omega^{2}}\right]^{\frac{1}{2}}$$

$$k_{+} = \omega\left[\frac{\mu\sigma^{2}}{4\varepsilon\omega^{2}}\right]^{\frac{1}{2}}$$

$$k_{+} = \omega\frac{\sigma}{2\omega}\left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}}$$

$$k_{+} = \frac{\sigma}{2}\left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}}$$

$$k_{+} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

Hence we have

$$k_{+} \cong \omega\sqrt{\varepsilon\mu} \quad \text{and} \quad k_{-} \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \qquad \text{--- 4.80}$$

The skin depth decreases if either the conductivity σ or the permeability μ increases. The skin depth is independent of frequency.

4.6 Poynting vector and Poynting's theorem

The work necessary to assemble a static charge distribution against the Coulomb repulsion of like charge is

$$W_E = \frac{\varepsilon_0}{2} \int_{Volume} E^2 d\tau \qquad \text{--- 4.81}$$

where E is the electric filed.

The work required to continue the current flow against the back emf is

$$W_B = \frac{1}{2\mu_0} \int_{Volume} B^2 d\tau \qquad \text{--- 4.82}$$

where B is the magnetic field.

Thus the total amount of energy stored in the field is

$$W_E + W_B = W_{EB} = \frac{\varepsilon_0}{2} \int_{Volume} E^2 d\tau + \frac{1}{2\mu_0} \int_{Volume} B^2 d\tau \qquad \text{--- 4.74}$$

$$W_{EB} = \frac{1}{2} \int_{Volume} \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \qquad \text{--- 4.75}$$

Now let us consider some charge and current configuration at time t, which produces fields E and B. In the next instant of time, the charge dq moved around a bit. Therefore according to the Lorentz force law the work done on an element of charge dq by the electromagnetic forces is

$$F \cdot dl = dq \ [E + (v \times B)] \cdot vdt \qquad [\because dl = vdt] \qquad \text{--- 4.76}$$

$$dw = dq \ [E + (v \times B)] \cdot vdt \qquad \text{--- 4.77}$$

$$dw = dq \ E \cdot vdt + dq \cdot (v \times B) \cdot vdt \qquad \text{--- 4.78}$$

Since the divergence of curl is zero, the second term in the right hand side of the equation 4.78 becomes zero. Thus we have

$$dw = dq \ E \cdot vdt \qquad \text{--- 4.79}$$

$$dw = E \cdot v \ dqdt \qquad \text{--- 4.80}$$

But $dq = \rho \ d\tau \qquad \left[\because \rho = \frac{dq}{d\tau} \right]$

Therefore $dw = E \cdot v \ \rho d\tau \ dt$, but $J = \rho v$

Hence $dw = E \cdot J \ d\tau \ dt$

or $$\frac{dw}{dt} = E \cdot J \; d\tau \qquad \text{--- 4.81}$$

Therefore the total work done on all the charges in some volume V is given by

$$\frac{dw}{dt} = \int_{Volume} (E \cdot J) \; d\tau \qquad \text{--- 4.82}$$

In the above equation 4.82, (E·J) is the work done per unit time per unit volume and is also known as the power delivered per unit volume. In other words $\int_{Volume} E \cdot J \; d\tau$ represents the rate of energy transferred into the electromagnetic field through the motion of free charges in volume V.

In equation 4.82, J can be eliminated by expressing the quantity E·J in terms of field alone by using Ampere's law with Maxwell's extra term.

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

Taking scalar product E on both sides, we get

$$E \cdot (\nabla \times B) = E \cdot (\mu_0 J) + E \cdot \mu_0 \varepsilon_0 \left(\frac{\partial E}{\partial t} \right)$$

$$E \cdot (\nabla \times B) = \mu_0 (E \cdot J) + \mu_0 \varepsilon_0 \left(E \cdot \frac{\partial E}{\partial t} \right)$$

or $$\mu_0 (E \cdot J) = E \cdot (\nabla \times B) - \mu_0 \varepsilon_0 \; E \cdot \frac{\partial E}{\partial t}$$

$$(E \cdot J) = \frac{1}{\mu_0} E \cdot (\nabla \times B) - \varepsilon_0 \; E \cdot \frac{\partial E}{\partial t} \qquad \text{--- 4.83}$$

But $$\nabla \cdot (E \times B) = B \cdot (\nabla \times E) - E \cdot (\nabla \times B)$$

$$E \cdot (\nabla \times B) = B \cdot (\nabla \times E) - \nabla \cdot (E \times B) \qquad \text{--- 4.84}$$

From Faraday's law we have

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

Now equation 4.84 becomes as

$$E \cdot (\nabla \times B) = B \cdot \left(-\frac{\partial B}{\partial t} \right) - \nabla \cdot (E \times B)$$

$$E \cdot (\nabla \times B) = -B \cdot \left(\frac{\partial B}{\partial t} \right) - \nabla \cdot (E \times B) \qquad \text{--- 4.85}$$

Substituting equation 4.85 in 4.83, we have

$$\left(E\cdot J\right)=\frac{1}{\mu_0}\left[-B\cdot\left(\frac{\partial B}{\partial t}\right)-\nabla\cdot\left(E\times B\right)\right]-\varepsilon_0 E\cdot\frac{\partial E}{\partial t}$$

$$\text{But } B\cdot\frac{\partial B}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left(B^2\right) \quad \text{and } E\cdot\frac{\partial E}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left(E^2\right)$$

Therefore

$$\left(E\cdot J\right)=\frac{1}{\mu_0}\left[-\frac{1}{2}\frac{\partial}{\partial t}\left(B^2\right)-\nabla\cdot\left(E\times B\right)\right]-\varepsilon_0\ \frac{1}{2}\frac{\partial}{\partial t}\left(E^2\right)$$

$$\left(E\cdot J\right)=-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\mu_0}B^2\right)-\frac{1}{\mu_0}\nabla\cdot\left(E\times B\right)-\frac{1}{2}\ \frac{\partial}{\partial t}\left(\varepsilon_0 E^2\right)$$

$$\left(E\cdot J\right)=-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\mu_0}B^2\right)-\frac{1}{2}\ \frac{\partial}{\partial t}\left(\varepsilon_0 E^2\right)-\frac{1}{\mu_0}\nabla\cdot\left(E\times B\right)$$

$$\left(E\cdot J\right)=-\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{\mu_0}B^2+\varepsilon_0 E^2\right]-\frac{1}{\mu_0}\nabla\cdot\left(E\times B\right) \qquad \text{--- 4.86}$$

Putting equation 4.86 in 4.82, we obtain

$$\frac{dw}{dt}=\int_{Volume}\left[-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\mu_0}B^2+\varepsilon_0 E^2\right)-\frac{1}{\mu_0}\nabla\cdot\left(E\times B\right)\right]d\tau \qquad \text{--- 4.87}$$

or

$$\frac{dw}{dt}=\int_{Volume}-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\mu_0}B^2+\varepsilon_0 E^2\right)d\tau-\frac{1}{\mu_0}\int_{Volume}\nabla\cdot\left(E\times B\right)\,d\tau \qquad \text{--- 4.88}$$

$$\frac{dw}{dt}=-\frac{d}{dt}\int_{Volume}\frac{1}{2}\left(\frac{1}{\mu_0}B^2+\varepsilon_0 E^2\right)d\tau-\frac{1}{\mu_0}\int_{Volume}\nabla\cdot\left(E\times B\right)\,d\tau \qquad \text{--- 4.89}$$

By applying Gauss's divergence theorem, the second integral term becomes

$$\frac{1}{\mu_0}\int_{Volume}\nabla\cdot\left(E\times B\right)\,d\tau=\int_{Surface}\left(E\times B\right)\,da$$

$$\frac{dw}{dt}=-\frac{d}{dt}\int_{Volume}\frac{1}{2}\left(\frac{1}{\mu_0}B^2+\varepsilon_0 E^2\right)d\tau-\frac{1}{\mu_0}\int_{Surface}\left(E\times B\right)\,da \qquad \text{--- 4.90}$$

This is known as Poynting theorem. It is also called as the work-energy theorem.

In equation 4.90, the first integral on the right represents the total energy stored in the fields. i.e. W_{EB}. The second integral represents the rate at which energy is carried out of V, across its boundary surface by the electromagnetic fields.

Generally Poynting theorem says that the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy which flowed out through the surface.

The energy per unit time, per unit area transported by the field is called the 'Poynting vector' S and is also known as 'energy flux density'. Hence

$$S = \frac{1}{\mu_0}(E \times B) \qquad \text{--- 4.91}$$

By using equations 4.75 and 4.91, Poynting theorem (i.e. equation 4.90) can be expressed more compactly as

$$\frac{dw}{dt} = -\frac{d}{dt} W_{EB} - \int_S S \cdot da \qquad \text{--- 4.92}$$

The work done W on the charges will increase their mechanical energy (i.e. kinetic, potential or whatsoever). Let U_M be the mechanical energy density, then

$$\frac{dw}{dt} = -\frac{d}{dt}(U_M \; d\tau) \qquad \text{--- 4.93}$$

Let U_{EB} is the energy density of the field, then

$$U_{EB} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_o} B^2\right) \qquad \text{--- 4.94}$$

$$\frac{dw_{EB}}{dt} = -\frac{d}{dt}\int_V U_{EB} \; d\tau \qquad \text{--- 4.95}$$

Putting equations 4.93 and 4.95 in 4.92, we obtain

$$\frac{d}{dt}\int_{Volume} (U_M \; d\tau) = -\frac{d}{dt}\int_{Volume} U_{EB} \; d\tau - \int_{Surface} S \cdot da$$

$$\frac{d}{dt}\int_{Volume} (U_M \; d\tau) + \frac{d}{dt}\int_{Volume} U_{EB} \; d\tau = -\int_{Surface} S \cdot da$$

$$\frac{d}{dt}\int_{Volume} (U_M + U_{EB}) d\tau = -\int_{Surface} S \cdot da$$

or

$$\frac{d}{dt}\int_{Volume} (U_M + U_{EB}) \; d\tau = -\int_{Volume} (\nabla \cdot S) \; da$$

or

$$(\nabla \cdot S) = -\frac{\partial}{\partial t}(U_M + U_{EB}) \qquad \text{--- 4.96}$$

This is the integral form of Poynting theorem.

Comparing it with the continuity equation, $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$, expressing the conservation of charge, the charge density is replaced by the total energy density (i.e. both mechanical and electrical) and the current density is replaced by the Poynting vector. S describes the flow of energy and J describes the flow of charge.

UNIT V : WAVES IN BOUNDED REGION AND RADIATION

Reflection and refraction of EM waves at the boundary of two conducting media – Normal incidence and oblique incidence – Brewster's angle– Wave guides – Rectangular wave guide – Cavity resonators – Radiation from an oscillating dipole –Transmission line theory – Transmission line as distribution circuit– Basic transmission line equations

5.1 Reflection and refraction of EM waves at the boundary of two conducting media

When the electromagnetic wave travelling in one medium smacks upon a second medium, the wave will be partially transmitted and partially reflected depending upon the constitutive parameters of media. Depending upon the manner in which the uniform plane is incident on the boundary, there are two classes of incidence.

(i) Normal incidence: When a uniform plane wave incidences normally to the boundary between the media, it is known as normal incidence.

(ii) Oblique incidence: When a uniform plane wave incidences obliquely to the boundary between the two media, then it is known as oblique incidence.

5.1.1 Normal incidence

When an electromagnetic wave travelling in one medium enters a second medium having different dielectric constant, permeability or conductivity, the wave will be partially transmitted and partially reflected mainly on account of discontinuity of the medium. In the case of a plane electromagnetic wave in air varying with time and incident normally upon the surface of a perfect conductor, the wave is entirely reflected because neither E nor B can exist within a perfect conductor. Since no energy is wasted in a perfect conductor, the amplitudes of E and B in the reflected wave are the same as in the incident wave. The only difference between the incident and reflected waves is in the direction of energy flow.

The boundary conditions used to analyse the reflection and refraction of electromagnetic wave through liner media are

(i) $\varepsilon_1 E_{1_\perp} - \varepsilon_2 E_{2_\perp} = \sigma_f$

(ii) $B_{1_\perp} = B_{2_\perp}$

(iii) $E_{1_{\parallel}} = E_{2_{\parallel}}$ and --- 5.1

(iii) $\frac{1}{\mu_1} B_{1_{\parallel}} - \frac{1}{\mu_2} B_{2_{\parallel}} = K_f \times \hat{n}$

where σ_f = free surface charge at the boundary,

K_f= free surface current at the boundary and

$\hat{n}$ is the unit vector perpendicular to the surface pointing from medium 2 to medium 1. For ohmic conductors, the free current at the boundary is zero (i.e. $K_f=0$).

Consider the YZ plane forms the boundary between a non-conducting linear medium 1 and conducting medium 2. A monochromatic plane wave, travelling in the x-direction and polarized in the y-direction, approaches from left to right as shown in Figure 5.1

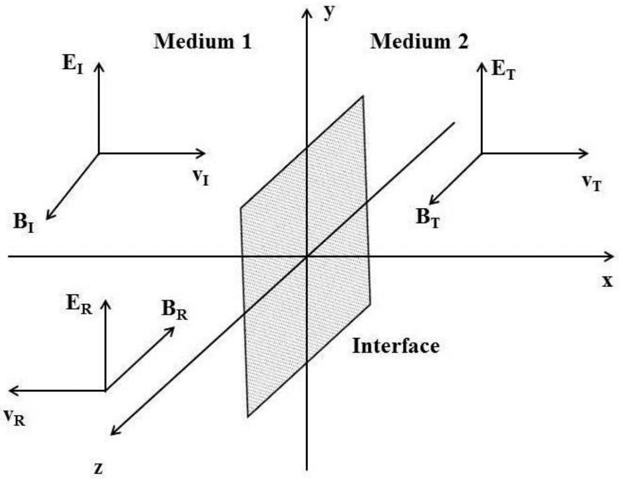


Figure 5.1

The electric and magnetic fields of the incident wave are given by

$$\tilde{E}_I(x,t)=\tilde{E}_{0_I}e^{i(k_1x-\omega t)}\hat{j} \quad \text{and} \quad \tilde{B}_I(x,t)=\frac{1}{v_1}\tilde{E}_{0_I}e^{i(k_1x-\omega t)}\tilde{k} \quad \left[\because B_{0_I}=\frac{1}{v_1}E_{0_I}\right]$$

This incident wave gives a reflected wave, which travels back to the left in medium 1. Hence the electric and magnetic fields of the reflected wave are

$$\tilde{E}_R(x,t)=\tilde{E}_{0_R}e^{i(-k_1x-\omega t)}\hat{j} \quad \text{and} \quad \tilde{B}_R(x,t)=-\frac{1}{v_1}\tilde{E}_{0_R}e^{i(-k_1x-\omega t)}\tilde{k} \qquad \text{--- 5.2}$$

The transmitted wave continues to the right in medium 2. Hence

$$\tilde{E}_T(x,t)=\tilde{E}_{0_T}e^{i(k_2x-\omega t)}\hat{j} \quad \text{and}$$

$$\tilde{B}_T(x,t)=\frac{1}{v_2}\tilde{E}_{0_T}e^{i(k_2x-\omega t)}\tilde{k}=\frac{k_2}{\omega}\tilde{E}_{0_T}e^{i(k_2x-\omega t)}\tilde{k}\left(\because v=\frac{\omega}{k}\right) \qquad \text{--- 5.3}$$

From equation 5.3, we observe that the transmitted wave is attenuated as it breaches into the conductor due to the complex nature of the wavenumber k. (k_2 has imaginary part in accordance with $k = k_+ + ik_-$).

At x = 0, $E_\perp = 0$ and $B_\perp = 0$

Since $E_\perp = 0$ on both sides, boundary condition 5.1 (i) yields $\sigma_f = 0\,[\because 0-0=0=\sigma_f]$.

Since $B_{\perp} = 0$, equation 5.1 (ii) is also equal to zero. But equation 5.1 (iii) gives

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T} \quad \text{--- 5.4}$$

and equation 5.1 (iv) becomes

$$\frac{1}{\mu_1} B_{1_{\parallel}} - \frac{1}{\mu_2} B_{2_{\parallel}} = 0 \; (\because K_f = 0) \quad \text{--- 5.5}$$

where μ_1 and μ_2 are the permeabilities of medium 1 and medium 2 respectively.

$$\frac{1}{\mu_1}\left[\frac{1}{v_1}\tilde{E}_{0_I} + \left(-\frac{1}{v_1}\tilde{E}_{0_R}\right)\right] - \frac{1}{\mu_2}\left(\frac{k_2}{\omega}\right)\tilde{E}_{0_T} = 0 \qquad \left[\because B_0 = \frac{1}{c}E_o\right] \quad \text{--- 5.6}$$

$$\frac{1}{\mu_1}\left[\frac{1}{v_1}\tilde{E}_{0_I} - \frac{1}{v_1}\tilde{E}_{0_R}\right] - \frac{1}{\mu_2}\left(\frac{k_2}{\omega}\right)\tilde{E}_{0_T} = 0 \quad \text{--- 5.7}$$

$$\frac{1}{\mu_1 v_1}\left[\tilde{E}_{0_I} - \tilde{E}_{0_R}\right] = \frac{1}{\mu_2}\left(\frac{k_2}{\omega}\right)\tilde{E}_{0_T} \quad \text{--- 5.8}$$

$$\frac{1}{\mu_1 v_1}\left[\tilde{E}_{0_I} - \tilde{E}_{0_R}\right] = \frac{k_2}{\mu_2 \omega}\tilde{E}_{0_T}$$

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \frac{\mu_1 v_1 k_2}{\mu_2 \omega}\left[\tilde{E}_{0_T}\right] \quad \text{--- 5.9}$$

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \left[\tilde{E}_{0_T}\right] \qquad \left[\because \beta = \frac{\mu_1 v_1 k_2}{\mu_2 \omega}\right] \quad \text{--- 5.10}$$

Adding equations 5.4 and 5.10, we get

$$2\tilde{E}_{0_I} = \tilde{E}_{0_T} + \beta\tilde{E}_{0_T} = \tilde{E}_{0_T}(1+\beta) \quad \text{--- 5.11}$$

$$\tilde{E}_{0_T} = \frac{2}{(1+\beta)}\tilde{E}_{0_I} \quad \text{--- 5.12}$$

Subtracting equation 5.10 from 5.4, we obtain

$$2\tilde{E}_{0_R} = \tilde{E}_{0_T} - \beta\tilde{E}_{0_T} = \tilde{E}_{0_T}(1-\beta) \quad \text{--- 5.13}$$

$$\tilde{E}_{0_R} = \frac{(1-\beta)}{2}\tilde{E}_{0_T} \quad \text{--- 5.14}$$

Putting equation 5.12 in 5.14, we have

$$\tilde{E}_{0_R} = \frac{(1-\beta)}{2} \times \frac{2}{(1+\beta)}\tilde{E}_{0_I} \quad \text{--- 5.15}$$

$$\tilde{E}_{0_R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0_I} \qquad \text{--- 5.16}$$

For a perfect conductor, β is equal to infinite (i.e. σ = ∞). Therefore we have

$$\tilde{E}_{0_R} = -\tilde{E}_{0_I} \qquad \text{--- 5.17}$$

$$\tilde{E}_{0_T} = 0 \qquad \text{--- 5.18}$$

Thus in the case of a perfect conductor, there is no transmitted wave. The incident wave is totally reflected with a 180° phase shift. That's why conductors such as silver make good mirrors.

5.1.2 Reflection and transmission of electromagnetic wave in a matter at obliqueincidence

When a plane electromagnetic wave is incident obliquely on the boundary, a part of the wave is transmitted and part of it is reflected. In this case the transmitted wave will be refracted i.e. the direction of propagation will be changed (Figure 5.2).

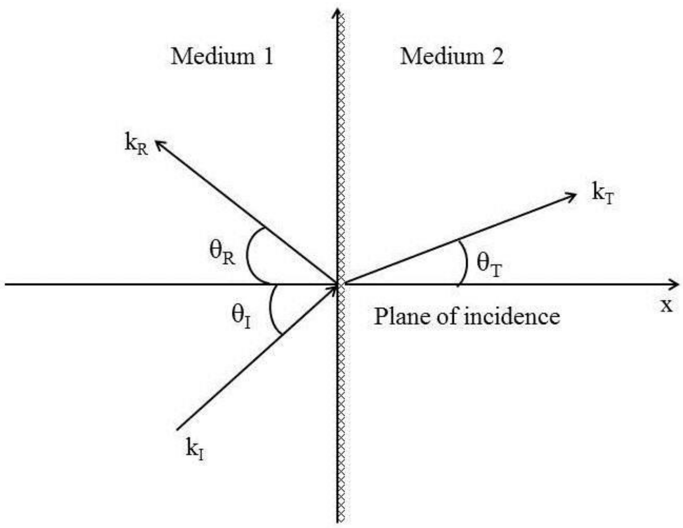


Figure 5.2

The electric and magnetic field vectors of the incident wave are given by

$$\tilde{E}_I(r,t) = \tilde{E}_{0_I} e^{i(k_I r - \omega t)} \quad \text{and} \quad \tilde{B}_I(r,t) = \frac{1}{v_1}[K_I \times E_I] \quad \left[\because B_{0_I} = \frac{1}{v_1} E_{o_I}\right] \qquad \text{---5.19}$$

This incident wave gives a reflected wave, which travels back to the left in medium 1. Hence the electric and magnetic fields of the reflected wave are

$$\tilde{E}_R(r,t) = \tilde{E}_{0_R} e^{i(k_R \cdot r - \omega t)} \quad \text{and} \quad \tilde{B}_R(r,t) = \frac{1}{v_1}[K_R \times E_R] \qquad \text{--- 5.20}$$

The transmitted wave continues to the right in medium 2. Hence

$$\tilde{E}_T(r,t) = \tilde{E}_{0T} e^{i(k_T \cdot r - \omega t)} \quad \text{and} \quad \tilde{B}_T(r,t) = \frac{1}{v_2}[K_T \times E_T] \qquad \text{--- 5.21}$$

According to the boundary conditions, the combined electric and magnetic fields in medium 1 is equal to the fields in medium 2. i.e., $E_I + E_R = B_I + B_R$.

The boundary conditions for the field vectors E and B are

(i) $\varepsilon_1 E_{1_\perp} = \varepsilon_2 E_{2_\perp}$

(ii) $B_{1_\perp} = B_{2_\perp}$

(iii) $E_{1_\parallel} = E_{2_\parallel}$ and

(iii) $\frac{1}{\mu_1} B_{1_\parallel} = \frac{1}{\mu_2} B_{2_\parallel}$

Hence applying the boundary conditions, the field equations become as

$$\varepsilon_1 \left[E_{0_I} + E_{0_R}\right]_x = \varepsilon_2 \left[E_{0_T}\right]_x \qquad \text{--- 5.22}$$

$$\left[B_{0_I} + B_{0_R}\right]_x = \left[B_{0_T}\right]_x \qquad \text{---5.23}$$

$$\left[E_{0_I} + E_{0_R}\right]_{y,z} = \left[E_{0_T}\right]_{y,z} \qquad \text{---5.24}$$

$$\frac{1}{\mu_1}\left[B_{0_I} + B_{0_R}\right]_{Y,Z} = \frac{1}{\mu_2}\left[B_{0_T}\right]_{Y,Z} \qquad \text{---5.25}$$

Assume that the incident wave is parallel to xy plane (Figure 5.3).

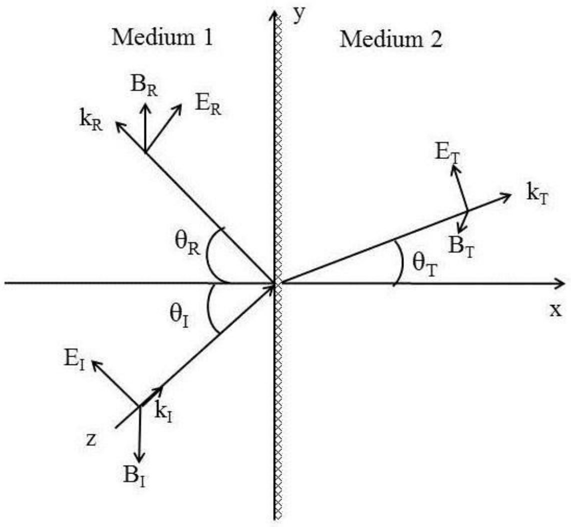


Figure 5.3

Now the equation 5.22 becomes as

$$\varepsilon_1\left[-E_{0_I}\sin\theta_I + E_{0_R}\sin\theta_R\right] = \varepsilon_2\left[-E_{0_T}\sin\theta_T\right] \qquad \text{--- 5.26}$$

where θ_I, θ_R and θ_T are the angle of incidence, reflection and transmission respectively.

Equation 5.23 becomes zero since the magnetic fields have no x components. From equation 5.24, we have

$$\left[E_{0_I}\cos\theta_I + E_{0_R}\cos\theta_R\right] = \left[E_{0_T}\cos\theta_T\right] \qquad \text{--- 5.27}$$

Since $B_0 = \frac{1}{v}E_0$, equation 5.25 can be rewritten as

$$\frac{1}{\mu_1 v_1}\left[E_{0_I} + E_{0_R}\right] = \frac{1}{\mu_2 v_2}\left[E_{0_T}\right] \qquad \text{--- 5.28}$$

$$\left[E_{0_I} - E_{0_R}\right] = \frac{\mu_1 v_1}{\mu_2 v_2}\left[E_{0_T}\right] = \beta E_{0_T} \quad \left[\because \beta = \frac{\mu_1 v_1}{\mu_2 v_2}\right] \qquad \text{--- 5.29}$$

when $\theta_I = \theta_R$, the equation 5.27 becomes

$$\left[E_{0_I}\cos\theta_I + E_{0_R}\cos\theta_I\right] = \left[E_{0_T}\cos\theta_T\right]$$

$$\left[E_{0_I} + E_{0_R}\right]\cos\theta_I = E_{0_T}\cos\theta_T$$

$$\left[E_{0_I} + E_{0_R}\right] = \frac{\cos\theta_T}{\cos\theta_I}\left[E_{0_T}\right] = \alpha E_{0_T} \quad \left[\because \alpha = \frac{\cos\theta_T}{\cos\theta_I}\right] \qquad \text{--- 5.30}$$

Adding equations 5.29 and 5.30, we get

$$\left[E_{0_I} - E_{0_R}\right] + \left[E_{0_I} + E_{0_R}\right] = \beta E_{0_T} + \alpha E_{0_T}$$

$$E_{0_I} - E_{0_R} + E_{0_I} + E_{0_R} = \beta E_{0_T} + \alpha E_{0_T}$$

$$2E_{0_I} = [\alpha + \beta]E_{0_T}$$

$$E_{0_T} = \frac{2}{[\alpha + \beta]}E_{0_I} \qquad \text{--- 5.31}$$

Subtracting equation 5.30 from 5.29, we get

$$-2E_{0_R} = (\beta - \alpha)E_{0_T}$$

$$2E_{0_R} = (\alpha - \beta)E_{0_T} \qquad \text{--- 5.32}$$

Substituting equation 5.31 in 5.32, we obtain

$$2E_{0_R} = (\alpha - \beta)\frac{2}{[\alpha + \beta]}E_{0_I}$$

$$E_{0_R} = \frac{(\alpha - \beta)}{2} \frac{2}{(\alpha + \beta)} E_{0_I}$$

$$E_{0_R} = \frac{(\alpha - \beta)}{(\alpha + \beta)} E_{0_I} \qquad \text{--- 5.33}$$

Equations 5.31 and 5.33 are known as *"Fresnel's equations"* for polarization in the plane of incidence.

If α > β, then the reflected wave is in phase with the incident wave and if α < β, the reflected wave is out of phase with the incident wave.

The amplitudes of the transmitted and reflected waves depend on the angle of incidence. This is because α is a function of θ_I.

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} \quad \left[\because \cos\theta = \sqrt{1-\sin^2\theta}\right] \qquad \text{--- 5.34}$$

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}}{\cos\theta_I} = \frac{\left[1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2\right]^{\frac{1}{2}}}{\cos\theta_I} \quad \left[\because \sin\theta_T = \frac{n_1}{n_2}\sin\theta_I\right] \qquad \text{--- 5.35}$$

Case (i) when the incidence is normal to the plane boundary, then $\theta_I = 0$. Hence, α=1.

Case (ii) at grazing incidence, i.e. $\theta_I = 90°$, α diverges and the wave is totally reflected.

Case (iii) at an intermediate angle, the reflected wave is completely extinguished. i.e. $E_{0_R} = 0$. This is possible only when α = β. Thus the angle of incidence at which the reflected wave is completely extinguished is known as *'Brewster's angle'*.

The reflection coefficient for waves polarized parallel to the plane of incidence is

$$R = \frac{I_R}{I_I} = \frac{\frac{1}{2}\varepsilon_1 v_1 E^2_{0_R} \cos\theta_R}{\frac{1}{2}\varepsilon_1 v_1 E^2_{0_I} \cos\theta_I} = \frac{E^2_{0_R}}{E^2_{0_I}} [\because \theta_I = \theta_R] \qquad \text{--- 5.36}$$

[The average power per unit area (i.e. intensity) I = $\frac{1}{2}\varepsilon v E_0^{\ 2}$]

$$R = \frac{E^2_{0_R}}{E^2_{0_I}} = \frac{(\alpha-\beta)^2}{[\alpha+\beta]^2} \quad [\text{From equation 5.33}] \qquad \text{--- 5.37}$$

Then the transmission coefficient is

$$T = \frac{I_T}{I_I} = \frac{\frac{1}{2}\varepsilon_2 v_2 E^2_{0_T} \cos\theta_T}{\frac{1}{2}\varepsilon_1 v_1 E^2_{0_I} \cos\theta_I}$$

$$T = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left[\frac{E_{0_T}}{E_{0_I}}\right]^2 \frac{\cos\theta_T}{\cos\theta_I} \quad \text{--- 5.38}$$

$$T = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left[\frac{E_{0_T}}{E_{0_I}}\right]^2 \alpha \quad \left[\because \alpha = \frac{\cos\theta_T}{\cos\theta_I}\right] \quad \text{--- 5.39}$$

$$T = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left[\frac{\left(\frac{2}{\alpha+\beta}\right)E_{0_I}}{E_{0_I}}\right]^2 \alpha \ (\text{Using equation 5.31}) \quad \text{--- 5.40}$$

$$T = \frac{\left(\frac{1}{\mu_2 v_2^{2}}\right)v_2}{\left(\frac{1}{\mu_1 v_1^{2}}\right)v_1}\left[\left(\frac{2}{\alpha+\beta}\right)\right]^2 \alpha \quad \left[\because \varepsilon = \frac{1}{\mu v^2}\right]$$

$$T = \frac{\left(\frac{1}{\mu_2 v_2}\right)}{\left(\frac{1}{\mu_1 v_1}\right)}\left[\left(\frac{2}{\alpha+\beta}\right)\right]^2 \alpha \quad \left[\because \varepsilon = \frac{1}{\mu v^2}\right]$$

$$T = \frac{\mu_1 v_1}{\mu_2 v_2}\left[\frac{2}{\alpha+\beta}\right]^2 \alpha$$

$$T = \beta\left[\frac{2}{\alpha+\beta}\right]^2 \alpha$$

$$T = \alpha\beta\left[\frac{2}{\alpha+\beta}\right]^2 \quad \text{--- 5.41}$$

According to the law of conservation of energy, R+T=1. At Brewster's angle, the reflection coefficient R becomes zero and the transmission coefficient T becomes 1, hence the law of conservation of energy is conserved. That is the energy per unit time reaching a particular spot of area on the surface is equal to the energy per unit time leaving the spot.

5.2 Brewster's angle

The angle of incidence at which the reflected wave is completely extinguished is known as *'Brewster's angle'.*

When $\alpha = \beta$, from equation 5.35, we have

$$\alpha = \frac{\left[1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2\right]^{\frac{1}{2}}}{\cos\theta_I} = \frac{\left[1-\left(\frac{n_1}{n_2}\sin\theta_B\right)^2\right]^{\frac{1}{2}}}{\cos\theta_B} = \beta \qquad \text{--- 5.42}$$

$$\left[1-\left(\frac{n_1}{n_2}\sin\theta_B\right)^2\right]^{\frac{1}{2}} = \beta\cos\theta_B$$

$$1-\left(\frac{n_1}{n_2}\sin\theta_B\right)^2 = \beta^2\cos^2\theta_B$$

$$1-\left(\frac{n_1}{n_2}\right)^2\left(\sin^2\theta_B\right) = \beta^2\cos^2\theta_B$$

$$\beta^2\left(1-\sin^2\theta_B\right) = 1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_B$$

$$\left(\beta^2-\beta^2\sin^2\theta_B\right) = 1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_I$$

$$\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_B = 1-\beta^2+\beta^2\sin^2\theta_B$$

$$\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_B - \beta^2\sin^2\theta_B = 1-\beta^2$$

$$\left[\left(\frac{n_1}{n_2}\right)^2-\beta^2\right]\sin^2\theta_B = 1-\beta^2$$

$$\sin^2\theta_B = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2-\beta^2} \qquad \text{--- 5.43}$$

If $\mu_1 = \mu_2$, then $\beta = \frac{n_2}{n_1}$. Now the equation 5.43 becomes

$$\sin^2\theta_B = \frac{1-\beta^2}{\left(\frac{1}{\beta}\right)^2-\beta^2} = \frac{1-\beta^2}{\frac{1}{\beta^2}-\beta^2} = \frac{\beta^2\left(1-\beta^2\right)}{1-\beta^4}$$

$$\sin^2\theta_B = \frac{\beta^2\left(1-\beta^2\right)}{\left(1+\beta^2\right)\left(1-\beta^2\right)} = \frac{\beta^2}{\left(1+\beta^2\right)} \qquad \text{--- 5.44}$$

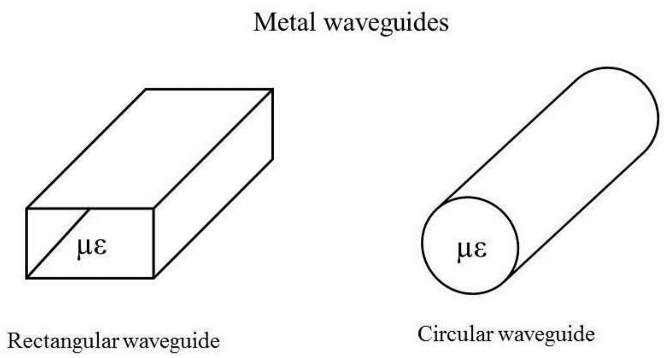
$$\text{We may deduce} \qquad \tan\theta_B = \beta = \frac{n_2}{n_1} \qquad \left[\because \sin^2 A = \frac{\tan^2 A}{(1+\tan^2 A)}\right] \qquad \text{--- 5.45}$$

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) \qquad \text{--- 5.46}$$

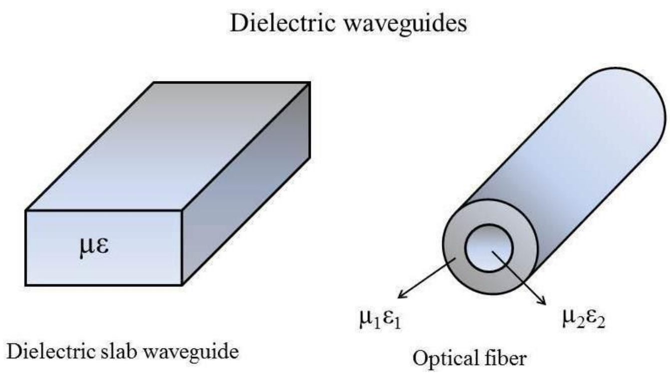
At this angle θ_B, which we call Brewster's angle, there is no reflected wave. Quartz windows of gas-laser in the gas discharge tube are inclined to the Brewster's angle to minimize reflection losses. Because of these windows the laser output is almost linearly polarized.

5.3 Wave guides

Waveguides, like transmission lines, are structures used to guideelectromagnetic wavesfrom point to point. However, the fundamentalcharacteristics of waveguide and transmission line waves (modes) are quitedifferent. The differences in these modes result from the basic differencesin geometry for a transmission line and a waveguide.Waveguides can be generally classified as either *metal waveguides* or*dielectric waveguides*(Figure 5.4).



(a)



(b)

Figure 5.4

Metal waveguides normally take the form of anenclosed conducting metal pipe. The waves propagating inside the metalwaveguide may be characterized by reflections from the conducting walls.The dielectric waveguide consists of dielectrics only and employsreflections from dielectric interfaces to propagate the electromagnetic wavealong the waveguide.

A wave guide is a hollow pipe of infinite length. Now let us discuss the propagation of electromagnetic waves along a hollow conducting pipe. Consider an electromagnetic wave confined to the interior of a hollow cylindrical pipe or wave guide. Here we assume that the wave guide is a perfect conductor. So that E=0 and B=0 inside the material itself. The boundary conditions at the inner walls of the waveguide are

$$(i) E_{ll} = 0 \text{ and } (ii) B_{\perp} = 0$$

From Maxwell's equations, we have

$$\left.\begin{aligned} &(i)\ \nabla \cdot E = 0 \\ &(ii)\ \nabla \cdot B = 0 \\ &(iii)\ \nabla \times E = -\frac{\partial B}{\partial t} \quad \text{and} \\ &(v)\ \nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} \end{aligned}\right\} \qquad \text{--- 5.47}$$

The confined waves are not transverse. In order to fit the boundary conditions, we have to include the longitudinal components of the fields E_x and B_x. Hence we have

$$E_0 = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} \quad ; B_0 = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

From Maxwell's equation 5.47 (iii)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times \left\{ \tilde{E}_0\, e^{i(kx-wt)} \right\} = -\frac{\partial}{\partial t}\left(B_0 e^{i(kx-wt)}\right)$$

$$\nabla \times \left\{ \left(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}\right) e^{i(kx-wt)} \right\} = -\frac{\partial}{\partial t}\left\{\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right) e^{i(kx-wt)}\right\}$$

$$\nabla \times \left\{ \left(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}\right) e^{i(kx-wt)} \right\} = -\hat{i}\frac{\partial B_x}{\partial t} e^{i(kx-wt)} - \hat{j}\frac{\partial B_y}{\partial t} e^{i(kx-wt)} - \hat{k}\frac{\partial B_z}{\partial t} e^{i(kx-wt)}$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x e^{i(kx-wt)} & E_y e^{i(kx-wt)} & E_z e^{i(kx-wt)} \end{vmatrix}$$

$$i\left\{\frac{\partial E_z}{\partial y}e^{i(kx-wt)}-\frac{\partial E_y}{\partial z}e^{i(kx-wt)}\right\}-j\left\{\frac{\partial E_z}{\partial x}e^{i(kx-wt)}-\frac{\partial E_x}{\partial z}e^{i(kx-wt)}\right\}+$$

$$k\left\{\frac{\partial E_y}{\partial x}e^{i(kx-wt)}-\frac{\partial E_x}{\partial y}e^{i(kx-wt)}\right\}=-\hat{i}\frac{\partial B_x}{\partial t}e^{i(kx-wt)}-\hat{j}\frac{\partial B_y}{\partial t}e^{i(kx-wt)}-\hat{k}\frac{\partial B_z}{\partial t}e^{i(kx-wt)}$$

The exponential factor in E is $e^{i(kx-\omega t)}$. Hence we have $(\partial/\partial x)$ =ik and $(\partial/\partial t)$ =-iω. Then

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=i\omega B_x \qquad \text{--- 5.48}$$

$$\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)=i\omega B_y \qquad \text{--- 5.49}$$

or $$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=i\omega B_y$$

or $$\frac{\partial E_x}{\partial z}-ikE_z=i\omega B_y$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z \qquad \text{--- 5.50}$$

or $$ikE_y-\frac{\partial E_x}{\partial y}=i\omega B_z \qquad \text{--- 5.51}$$

Similarly using Maxwell's equation 5.47 (iv), we have

$$\nabla\times\left\{\tilde{B}_0 e^{i(kx-wt)}\right\}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(E_0 e^{i(kx-wt)}\right)$$

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=-\frac{i\omega}{c^2}E_x \qquad \text{--- 5.52}$$

$$\frac{\partial B_x}{\partial z}-ikB_z=-\frac{i\omega}{c^2}E_y \qquad \text{--- 5.53}$$

$$ikB_y-\frac{\partial B_x}{\partial y}=-\frac{i\omega}{c^2}E_z \qquad \text{--- 5.54}$$

From equation 5.53, we have

$$ikB_z=\frac{\partial B_x}{\partial z}+\frac{i\omega}{c^2}E_y$$

$$B_z=\frac{1}{ik}\left[\frac{\partial B_x}{\partial z}+\frac{i\omega}{c^2}E_y\right] \qquad \text{--- 5.55}$$

Substituting equation 5.55 in 5.51

$$ikE_y - \frac{\partial E_x}{\partial y} = i\omega \frac{1}{ik}\left[\frac{\partial B_x}{\partial z} + \frac{i\omega}{c^2}E_y\right]$$

$$ikE_y - \frac{\partial E_x}{\partial y} = \frac{\omega}{k}\left[\frac{\partial B_x}{\partial z} + \frac{i\omega}{c^2}E_y\right]$$

$$ikE_y - \frac{\partial E_x}{\partial y} = \frac{\omega}{k}\frac{\partial B_x}{\partial z} + \frac{i\omega^2}{kc^2}E_y$$

$$ikE_y - \frac{i\omega^2}{kc^2}E_y = \frac{\omega}{k}\frac{\partial B_x}{\partial z} + \frac{\partial E_x}{\partial y}$$

Multiply throughout by ik, then

$$-k^2E_y + \frac{\omega^2}{c^2}E_y = i\omega\frac{\partial B_x}{\partial z} + ik\frac{\partial E_x}{\partial y}$$

$$-k^2E_y + \frac{\omega^2}{c^2}E_y = ik\frac{\partial E_x}{\partial y} + i\omega\frac{\partial B_x}{\partial z}$$

$$\frac{\omega^2}{c^2}E - k^2E_{y_y} = ik\frac{\partial E_x}{\partial y} + i\omega\frac{\partial B_x}{\partial z}$$

$$E_y\left[\frac{\omega^2}{c^2} - k^2\right] = i\left[k\frac{\partial E_x}{\partial y} + \omega\frac{\partial B_x}{\partial z}\right]$$

$$E_y = \frac{i}{\left[\frac{\omega^2}{c^2} - k^2\right]}\left[k\frac{\partial E_x}{\partial y} + \omega\frac{\partial B_x}{\partial z}\right] \qquad \text{--- 5.55a}$$

Similarly we can obtain E_z, B_y and B_z

$$E_z = \frac{i}{\left[\frac{\omega^2}{c^2} - k^2\right]}\left[k\frac{\partial E_x}{\partial z} + \omega\frac{\partial B_x}{\partial y}\right] \qquad \text{--- 5.55b}$$

$$B_y = \frac{i}{\left[\frac{\omega^2}{c^2} - k^2\right]}\left[k\frac{\partial B_x}{\partial y} - \frac{\omega}{c^2}\frac{\partial E_x}{\partial z}\right] \qquad \text{--- 5.55c}$$

$$B_z = \frac{i}{\left[\frac{\omega^2}{c^2} - k^2\right]}\left[k\frac{\partial B_x}{\partial z} + \frac{\omega}{c^2}\frac{\partial E_x}{\partial y}\right] \qquad \text{--- 5.55d}$$

Then it is very easy to find the longitudinal components E_x and B_x. From equation 5.48, we have

$$i\omega B_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

$$B_x = \frac{1}{i\omega}\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right]$$

Putting the values of E_z (equation 5.55b) and E_y (equation 5.55a), we get

$$B_x = \frac{1}{i\omega}\left\{\frac{\partial}{\partial y}\left[\frac{i}{\frac{\omega^2}{c^2} - k^2}\left[k\frac{\partial E_x}{\partial z} + \omega\frac{\partial B_x}{\partial y}\right]\right]\right\} - \left\{\frac{\partial}{\partial z}\left[\frac{i}{\left[\frac{\omega^2}{c^2} - k^2\right]}\left[k\frac{\partial E_x}{\partial y} + \omega\frac{\partial B_x}{\partial z}\right]\right]\right\}$$

$$B_x = \left(\frac{1}{i\omega}\right)\left(\frac{i}{\frac{\omega^2}{c^2} - k^2}\right)\left\{k\frac{\partial^2 E_x}{\partial y \partial z} - \omega\frac{\partial^2 B_x}{\partial y^2} - k\frac{\partial^2 E_x}{\partial z \partial y} - \omega\frac{\partial^2 B_x}{\partial z^2}\right\}$$

$$B_x = \left(\frac{-\omega}{\omega\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]}\right)\left\{\frac{\partial^2 B_x}{\partial y^2} - \frac{\partial^2 B_x}{\partial z^2}\right\}$$

$$B_x\left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = -\left\{\frac{\partial^2 B_x}{\partial y^2} - \frac{\partial^2 B_x}{\partial z^2}\right\}$$

$$\frac{\partial^2 B_x}{\partial y^2} - \frac{\partial^2 B_x}{\partial z^2} + B_x\left[\frac{\omega^2}{c^2} - k^2\right] = 0 \qquad \text{---5.56}$$

Similarly for E_x

$$\frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_x}{\partial z^2} + E_x\left[\frac{\omega^2}{c^2} - k^2\right] = 0 \qquad \text{---5.57}$$

From equation 5.56, we obtain

$$\left[\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{\omega^2}{c^2} - k^2\right)\right] B_x = 0 \qquad \text{--- 5.58}$$

From equation 5.57, we obtain

$$\left[\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \left(\frac{\omega^2}{c^2} - k^2\right)\right] E_x = 0 \qquad \text{--- 5.59}$$

5.3.2 Mode Classification

In uniform waveguides it is common to classify the various wave solutions found from the previous analysisinto the following types:

(i) TEM waves: Waves with no electric or magnetic field in the direction of propagation ($B_x = E_x = 0$).Plane waves and transmission-line waves are common examples.

(ii) TM waves: Waves with an electric field but no magnetic field in the direction of propagation ($B_x = 0, E_x \neq 0$). These are sometimes referred to as E waves.

(iii) TE waves: Waves with a magnetic field but no electric field in the direction of propagation ($B_x \neq 0, E_x = 0$). These are sometimes referred to as B waves.

(iv) Hybrid waves: Sometimes the boundary conditions require all field components. These waves can beconsidered as a coupling of TE and TM modes by the boundary.

5.4 Rectangular waveguides

Rectangular waveguides (Figure 5.5) are the one of the earliest type of the transmission lines. They are used in manyapplications. A lot of components such as isolators, detectors, attenuators, couplers and slotted lines areavailable for various standard waveguide bands between 1 GHz to above 220 GHz.A rectangular waveguide supports TM and TE modes but not TEM waves because we cannot define a uniquevoltage since there is only one conductor in a rectangular waveguide. The shape of a rectangular waveguideis as shown below. A material with permittivity ε and permeability μ fills the inside of the conductor.A rectangular waveguide cannot propagate below somecertain frequency. This frequency is called the *cut-offfrequency*.

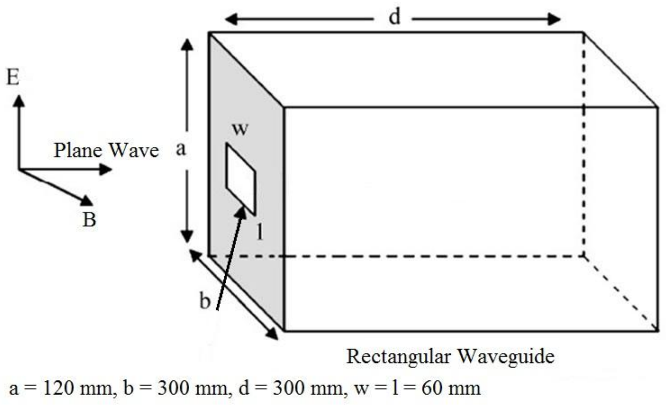


Figure 5.5

5.5 Cavity resonators

A cavity is known as a space totally enclosed by a metallic conductor and excited in such a way that it becomes a source of electromagnetic oscillations. The different types of cavities are: microwave cavity, microwave resonance cavity, resonant cavity, resonant chamber, resonant element, rhumbatron, tuned cavity and waveguide resonator.

Cavity Resonator is an oscillatory system that operates at super high frequencies.It is the analogy of an oscillatory circuit. The cavity resonator has the form of a volume filled with a dielectric-air, in most cases. The volume is bounded by a conducting surface or by a space having differing electrical or magnetic properties. Hollow cavity resonators i.e. cavities enclosed by metal walls-are most widely used. Generally, the boundary surface of a cavity resonator can have an arbitrary shape. However,in practice, only a few very simple shapes are used because such shapes simplify the configuration of the electromagnetic field, the design and manufacture of resonators. These shapes include right circular cylinders, rectangular parallelepipeds, toroids, and spheres. It is convenient to regard some types of cavity resonators as sections of hollow or dielectric wave guides limited by two parallel planes.

The solution of the problem of the natural (or normal) modes of oscillation of the electromagnetic field in a cavity resonator reduces to the solution of Maxwell's equations with appropriate boundary conditions. The process of storing electromagnetic energy in a cavity resonator can be clarified by the following example: if a plane wave is in some way excited between two parallel reflecting planes such that the wave propagation is perpendicular to the planes, then when the wave arrives at one of the planes, it will be totally reflected. Multiple reflection from the two planes produces waves that propagate in opposite directions and interfere with each other. If the distance between the planes is $L = n\lambda/2$, where λ is the wavelength and n is an integer, then the interference of the waves will produce a standing wave (Figure 5.6).The amplitude of this wave will increase rapidly if multiple reflections are present. Electromagnetic energy will be stored in the space between the planes. This effect is similar to the resonance effect in an oscillatory circuit.

5.5.1. Quality of the resonator

Normal oscillations can exist in a cavity resonator for an infinitely long time if there are no energy losses. However, in practice, energy losses in a cavity resonator are unavoidable. The alternating magnetic field induces electric currents on the inside walls of the resonator, which heat the walls and thus cause energy losses (conduction losses). Moreover, if there are apertures in the walls of the cavity and if these apertures intersect the lines of current, then an electromagnetic field will be generated outside the cavity, which causes energy losses by radiation. In addition, there are energy losses within the dielectric and losses caused by coupling with external circuits. The ratio of energy that is stored in a cavity resonator to the total losses in the resonator taken over one oscillation is called the *'figure of merit', or 'quality factor', or 'Q',* of the cavity resonator. The higher the figure of merit, the better is the quality of the resonator.

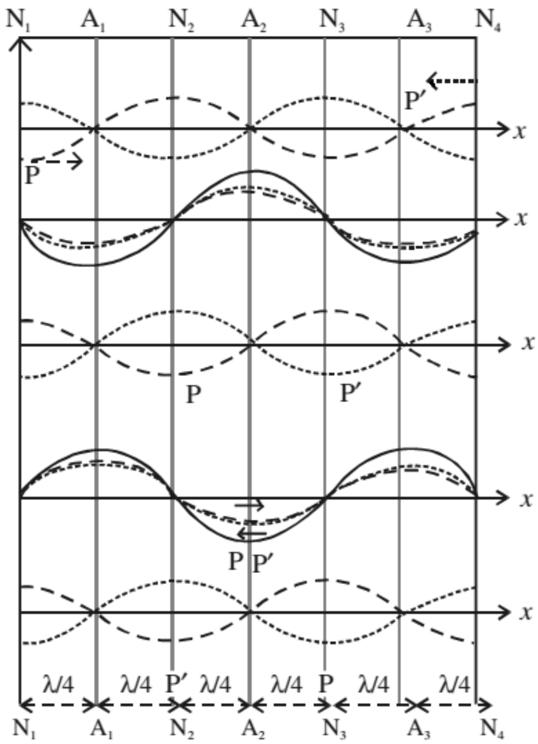


Figure 5.6

By analogy with wave guides, the oscillations that occur in a cavity resonator are classified in groups. In this classification, the grouping depends on the presence or absence of axial and radial (transverse) components in the spatial distribution of the electromagnetic field. Oscillations of the B (or TE) type have an axial component in the magnetic field only; oscillations of the E (or TM) type have an axial component in the electric field only. Finally, oscillations of the TEM type do not have axial components in either the electric or the magnetic field. An example of a cavity resonator in which TEM oscillations can be excited is the cavity between two conducting coaxial cylinders having end boundaries that are formed by plane conducting walls perpendicular to the axis of the cylinders.

5.6 Radiation from an oscillating dipole

Consider two metal spheres separated by a distance s and connected by a wire as shown in Figure 5.7.

Let q(t) be the charge on the upper sphere and –q (t) be the charge on the lower sphere in time t. Also assume that the two charges are driven to move back and forth from one end to the other at a frequency ω. Therefore

$$q(t) = q_0 \cos \omega t \qquad \text{--- 5.47}$$

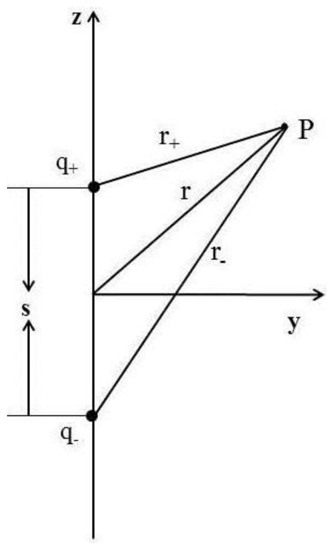


Figure 5.7

An oscillating electric dipole is described as

$$P(t) = q(t)s \qquad \text{--- 5.48}$$

$$P(t) = q_0 \cos \omega t\, s$$

$$P(t) = (q_0 s) \cos \omega t$$

or

$$P(t) = p_0 \cos \omega t\, \hat{k} \quad [\because q_0 s = p_0] \qquad \text{--- 5.49}$$

where p_0 is the maximum value of the dipole moment.

The retarded potential at P is given by

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_0 \cos \omega\left(t - \frac{r_+}{c}\right)}{r_+} - \frac{q_0 \cos \omega\left(t - \frac{r_-}{c}\right)}{r_-}\right] \qquad \text{--- 5.50}$$

Using law of cosines, we may have

$$r_{\pm} = \sqrt{r^2 \mp rs\cos\theta + \left(\frac{s}{2}\right)}^{2} \qquad \text{--- 5.51}$$

$$r_{\pm} = \sqrt{r^2\left(1 \mp \frac{rs}{r^2}\cos\theta\right)} \qquad [\because s << r]$$

$$r_{\pm} = \sqrt{r^2\left(1 \mp \frac{s}{r}\cos\theta\right)}$$

$$r_{\pm} = \left[r^2\left(1 \mp \frac{s}{r}\cos\theta\right)\right]^{\frac{1}{2}}$$

$$r_{\pm} = \left[r\left(1 \mp \frac{s}{r}\cos\theta\right)^{\frac{1}{2}}\right]$$

$$r_{\pm} = \left[r\left(1 \mp \frac{1}{2}\frac{s}{r}\cos\theta\right)\right] \qquad \text{--- 5.52}$$

$$\frac{1}{r_{\pm}} = \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{s}{r}\cos\theta\right) \qquad \text{--- 5.53}$$

Therefore

$$\cos\omega\left(t - \frac{r_{\pm}}{c}\right) = \cos\left[\omega\left\{t - \frac{1}{c}\cdot r\left(1 \mp \frac{1}{2}\frac{s}{r}\cos\theta\right)\right\}\right] \qquad \text{---5.54}$$

$$\cos\omega\left(t - \frac{r_{\pm}}{c}\right) = \cos\left[\omega\left\{t - \frac{r}{c} \pm \frac{1}{2}\frac{s}{c}\cos\theta\right\}\right]$$

$$\cos\omega\left(t - \frac{r_{\pm}}{c}\right) = \cos\left[\omega\left(t - \frac{r}{c}\right) \pm \frac{1}{2}\frac{\omega s}{c}\cos\theta\right] \qquad \text{---5.55}$$

But

$$\cos\left[\omega\left(t - \frac{r}{c}\right) \pm \frac{\omega s}{2c}\cos\theta\right] = \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega s}{2c}\cos\theta\right) \mp \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega s}{2c}\cos\theta\right) \qquad \text{--- 5.56}$$

$$[\text{From } \cos(A+B) = \cos A\cos B - \sin A\sin B]$$

For perfect dipole, $s << r$

$$\cos\left(\frac{\omega s}{2c}\cos\theta\right) = 1, \quad [\cos(0) = 1, \ \because \theta \text{ is very small}]$$

and $$\sin\left(\frac{\omega s}{2c}\cos\theta\right) = \frac{\omega s}{2c}\cos\theta \quad [\sin\theta = \theta, \because \theta \text{ is very small}]$$

Now equation 5.56 becomes as

$$\cos\left[\omega\left(t-\frac{r}{c}\right)\pm\frac{\omega s}{2c}\cos\theta\right] = \cos\omega\left(t-\frac{r}{c}\right) \mp \sin\omega\left(t-\frac{r}{c}\right)\frac{\omega s}{2c}\cos\theta \qquad \text{--- 5.57}$$

Substituting equation 5.57 in 5.55, we obtain

$$\cos\omega\left(t-\frac{r_{\pm}}{c}\right) = \cos\omega\left(t-\frac{r}{c}\right) \mp \frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right)$$

Substituting equations 5.53 and 5.57 in 5.50, we get

$$V(r,\theta,t) = \frac{q_0}{4\pi\varepsilon_0}\left[\begin{array}{l}\left\{\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right)\right\}\frac{1}{r}\left(1+\frac{1}{2}\frac{s}{r}\cos\theta\right) - \\ \left\{\cos\omega\left(t-\frac{r}{c}\right) + \frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right)\right\}\frac{1}{r}\left(1-\frac{1}{2}\frac{s}{r}\cos\theta\right)\end{array}\right] \qquad \text{--- 5.58}$$

$$V(r,\theta,t) = \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left[\begin{array}{r}\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{2}\frac{s}{r}\cos\theta\cos\omega\left(t-\frac{r}{c}\right) \\ -\frac{\omega s^2}{4rc}\cos^2\theta\,\sin\omega\left(t-\frac{r}{c}\right) - \cos\omega\left(t-\frac{r}{c}\right) \\ -\frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{2}\frac{s}{r}\cos\theta\cos\omega\left(t-\frac{r}{c}\right) \\ +\frac{\omega s^2}{4rc}\cos^2\theta\,\sin\omega\left(t-\frac{r}{c}\right)\end{array}\right]$$

$$V(r,\theta,t) = \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}\left[-2\frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right) + 2\frac{1}{2}\frac{s}{r}\cos\theta\cos\omega\left(t-\frac{r}{c}\right)\right]$$

$$V(r,\theta,t) = \frac{q_0}{4\pi\varepsilon_0}\left(\frac{1}{r}\right)\left[-\frac{\omega s}{c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right) + \frac{s}{r}\cos\theta\cos\omega\left(t-\frac{r}{c}\right)\right]$$

$$V(r,\theta,t) = \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r}s\cos\theta\left[-\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\right]$$

$$V(r,\theta,t) = \frac{(q_0 s)\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\right]$$

$$V(r,\theta,t) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\right] \qquad \text{--- 5.59}$$

In the static limit i.e. $\omega = 0$

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left[-(0) + \frac{1}{r}\ (1)\right]$$

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r^2} \qquad \text{--- 5.60}$$

The equation 5.60 gives the potential of an oscillating electric dipole.

In radiation zone, i.e. the field at large distance from the source i.e. $r >> \frac{c}{\omega}$, the second term in equation 5.59 vanishes.

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\ \sin\omega\left(t - \frac{r}{c}\right)\right]$$

$$V(r,\theta,t) = \frac{p_0 \omega}{4\pi\varepsilon_0 c}\left[\left(\frac{\cos\theta}{r}\right)\left\{-\sin\omega\left(t - \frac{r}{c}\right)\right\}\right]$$

$$V(r,\theta,t) = -\frac{p_0 \omega}{4\pi\varepsilon_0 c}\left[\left(\frac{\cos\theta}{r}\right)\left\{\sin\omega\left(t - \frac{r}{c}\right)\right\}\right] \text{--- 5.61}$$

The vector potential is determined by the current flowing through the wire. Hence we write

$$I(t) = \frac{dq}{dt}\hat{k}$$

$$I(t) = \frac{dq}{dt}(q_0\ \cos\omega t)\hat{k}$$

$$I(t) = -q_0\ \omega \sin\omega t\ \hat{k}$$

Therefore the vector potential due to an oscillating dipole is

$$A(r,t) = \frac{\mu_0}{4\pi}\int_{s/2}^{s/2}\left[-\frac{q_0\ \omega\sin\omega\left(t - \frac{R}{c}\right)\hat{k}}{r} dz\ \ [\text{From Figure 5.8}]\right] \qquad \text{--- 5.62}$$

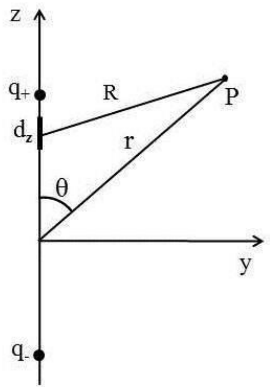


Figure 5.8

The integration itself introduces a factor s. Hence, we can replace the integrand by its value at the centre. i.e.

$$A(r,\theta,t) = -\frac{\mu_0 q_0\ \omega}{4\pi r}\ \sin\omega\left(t-\frac{r}{c}\right)\left[\ \frac{s}{2}-\left(-\frac{s}{2}\right)\ \right]\hat{k}$$

$$A(r,\theta,t) = -\frac{\mu_0 q_0\ \omega}{4\pi r}\ \sin\omega\left(t-\frac{r}{c}\right)[s]\,\hat{k}$$

$$A(r,\theta,t) = -\frac{\mu_0 q_0 s\ \omega}{4\pi r}\ \sin\omega\left(t-\frac{r}{c}\right)\ \hat{k}$$

$$A(r,\theta,t) = -\frac{\mu_0 p_0\ \omega}{4\pi r}\ \sin\omega\left(t-\frac{r}{c}\right)\ \hat{k}\quad (q_0 s = p_0) \qquad \text{--- 5.63}$$

For spherical coordinates

$$\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

Hence equation 5.63 becomes

$$A(r,\theta,t) = -\frac{\mu_0 p_0\ \omega}{4\pi r}\ \sin\omega\left(t-\frac{r}{c}\right)\ \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) \qquad \text{--- 5.64}$$

From the scalar potential equation we have,

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} \qquad \text{--- 5.65}$$

Substituting equation 5.61 in 5.65, we get

$$\nabla V = \frac{\partial}{\partial r}\left\{-\frac{p_0\omega}{4\pi\varepsilon_0 c}\ \left(\frac{\cos\theta}{r}\right)\sin\omega\left(t-\frac{r}{c}\right)\right\}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left\{-\frac{p_0\omega}{4\pi\varepsilon_0 c}\ \left(\frac{\cos\theta}{r}\right) - \sin\omega\left(t-\frac{r}{c}\right)\right\}\hat{\theta} \qquad \text{--- 5.66}$$

$$\nabla V = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\cos\theta\left\{\left(-\frac{1}{r^2}\right)\sin\omega\left(t-\frac{r}{c}\right)+\frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\left(-\frac{\omega}{c}\right)\right\}\hat{r}+ \frac{1}{r^2}\left(-\frac{p_0\omega}{4\pi\varepsilon_0 c}\right)\left\{\sin\omega\left(t-\frac{r}{c}\right)(-\sin\theta)\right\}\hat{\theta} \quad \text{--- 5.67}$$

Since $r >> \frac{c}{\omega}$, then r^2 is very large and therefore $\frac{1}{r^2}$ can be neglected. Hence equation 5.67 becomes as

$$\nabla V = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\cos\theta\left\{\frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\left(-\frac{\omega}{c}\right)\right\}\hat{r}$$

$$\nabla V = \frac{p_0\omega^2}{4\pi\varepsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{r} \quad \text{--- 5.68}$$

Then from equation 5.65, we have

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t}\left[-\frac{\mu_0 p_0\,\omega}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\left(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}\right)\right]$$

$$\frac{\partial A}{\partial t} = -\frac{\mu_0 p_0\,\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\left(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}\right) \quad \text{--- 5.69}$$

The relation between the electric field, scalar potential and vector potential is given by

$$E = -\nabla V - \frac{\partial A}{\partial t} \quad \text{--- 5.70}$$

Substituting equations 5.68 and 5.69 in 5.70, we get

$$E = -\frac{p_0\omega^2}{4\pi\varepsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{r} + \frac{\mu_0 p_0\,\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\left(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}\right) \quad \text{--- 5.71}$$

We know $c^2 = \frac{1}{\mu_0\varepsilon_0}$ and therefore $\frac{p_0\omega^2}{4\pi\varepsilon_0\frac{1}{\mu_0\varepsilon_0}} = \frac{\mu_0 p_0\omega^2}{4\pi}$

Now equation 5.71 can be rewritten as

$$E = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos\theta\,\hat{r}\cos\omega\left(t-\frac{r}{c}\right) + \frac{\mu_0 p_0\,\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\cos\theta\,\hat{r} - \frac{\mu_0 p_0\,\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\sin\theta\,\hat{\theta} \quad \text{--- 5.72}$$

$$E = -\frac{\mu_0 p_0\,\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\theta} \quad \text{--- 5.73}$$

Then $$\nabla\times A=\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}$$ --- 5.74

Putting equation 5.64 in 5.74, we obtain

$$\nabla\times A=-\frac{\mu_0 p_0\ \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)+\frac{\sin\theta}{r}\ \sin\omega\left(t-\frac{r}{c}\right)\hat{\varphi}\right\}$$ --- 5.75

Since r >> $\frac{c}{\omega}$, the second term in equation 5.75 vanishes. Hence

$$\nabla\times A=-\frac{\mu_0 p_0\ \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)\hat{\varphi}\right\}$$ --- 5.76

But we know B = ∇×A. Therefore the magnetic field B is given by

$$B=-\frac{\mu_0 p_0\ \omega^2}{4\pi rc}\ \sin\theta\cos\omega\left(t-\frac{r}{c}\right)\hat{\varphi}$$

or $$B=-\frac{\mu_0 p_0\ \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\varphi}$$ --- 5.77

Equations 5.73 and 5.77 are the electric and magnetic field vectors (E and B respectively) of monochromatic waves of frequency ω propagating in the radial direction with a velocity of light. E and B are also in phase, transverse and mutually perpendicular. The ratio of their amplitudes of the field vectors is equal to the velocity of light i.e. $\frac{E^0}{B_0}=C$.

The energy radiated by an oscillating electric dipole is determined by Poynting vector S.

$$S=\frac{1}{\mu_0}\,(E\times B)$$ --- 5.78

Therefore

$$S=\frac{1}{\mu_0}\left\{-\frac{\mu_0 p_0\,\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\theta}\ \times-\frac{\mu_0 p_0\ \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\varphi}\right\}$$

$$S=\frac{1}{\mu_0}\frac{\mu_0{}^2}{c}\left\{\frac{p_0\ \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{r}\right\}^2 \quad \left[\because\ \hat{\theta}\hat{\varphi}=\hat{r}\right]$$

$$S=\frac{\mu_0}{c}\left\{\frac{p_0\ \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{r}\right\}^2$$ --- 5.79

The intensity of the wave is obtained by taking the time average of equation 5.79 over a complete cycle. Hence we obtain

$$S = \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{16\pi^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\omega\left(t-\frac{r}{c}\right)\hat{r}$$

$$\langle S\rangle = \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{16\pi^2}\left(\frac{\sin^2\theta}{r^2}\right)\cdot\frac{1}{2}\ \hat{r}\quad\left(\langle\cos^2\theta\rangle = \frac{1}{2}\right)$$

$$\langle S\rangle = \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{32\pi^2}\left(\frac{\sin^2\theta}{r^2}\right)\cdot\hat{r} \qquad \text{--- 5.80}$$

Along the axis of the dipole sin θ = 0. Hence there is no radiation along the direction of axis of the dipole. Therefore the profile of the intensity of radiation seems to be a donut with its maximum in the equatorial plane (Figure 5.9).

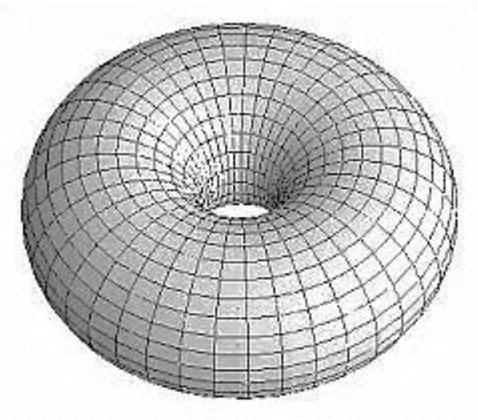


Figure 5.9

The total power radiated by the oscillating electric dipole is found by integrating S over a sphere of radius r. Therefore

$$\langle P\rangle = \int \langle S\rangle\cdot da \qquad \text{--- 5.81}$$

$$\langle P\rangle = \int \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{32\pi^2}\left(\frac{\sin^2\theta}{r^2}\right)\cdot da$$

$$\langle P\rangle = \int \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{32\pi^2}\left(\frac{\sin^2\theta}{r^2}\right)\cdot r^2\sin\theta\, d\theta\, d\phi$$

$$\langle P\rangle = \int \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{32\pi^2}\ \sin^2\theta\cdot\sin\theta\, d\theta\, d\phi$$

$$\langle P\rangle = \int \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{32\pi^2}\ \sin^3\theta\, d\theta\, d\phi$$

$$\langle P\rangle = \frac{\mu_0}{c}\frac{p_0^{\ 2}\ \omega^4}{32\pi^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin^3\theta\, d\theta\, d\phi$$

$$\langle P\rangle = \frac{\mu_0}{c}\frac{p_0^{\,2}\,\omega^4}{32\pi^2}\int_{\theta=0}^{\pi}\sin^3\theta\, d\theta\int_{\phi=0}^{2\pi}d\phi$$

$$\langle P\rangle = \frac{\mu_0}{c}\frac{p_0^{\,2}\,\omega^4}{32\pi^2}\,\frac{4}{3}\,2\pi$$

$$\langle P\rangle = \frac{\mu_0}{c}\frac{p_0^{\,2}\,\omega^4}{32\pi}\,\frac{8}{3}$$

$$\langle P\rangle = \frac{\mu_0}{c}\frac{p_0^{\,2}\,\omega^4}{4\pi}\,\frac{1}{3}$$

$$\langle P\rangle = \frac{\mu_0}{c}\frac{p_0^{\,2}\,\omega^4}{4\pi}\,\frac{1}{3}\,\frac{c^2}{c^2}$$

$$\langle P\rangle = \frac{\mu_0}{c}\frac{p_0^{\,2}\,\omega^4}{4\pi c^2}\,\frac{1}{3}\,\frac{1}{\mu_0\varepsilon_0}$$

$$\langle P\rangle = \frac{p_0^{\,2}\,\omega^4}{4\pi c^3}\,\frac{1}{3}\,\frac{1}{\varepsilon_0} \quad = \quad \frac{p_0^{\,2}\,\omega^4}{4\pi\varepsilon_0}\,\frac{1}{3c^3}$$

$$\langle P\rangle = \frac{p_0^{\,2}\,\omega^4}{4\pi\varepsilon_0}\,\frac{1}{3c^3}$$

$$\langle P\rangle = \frac{1}{4\pi\varepsilon_0}\frac{p_0^{\,2}\,\omega^4}{3c^3} \qquad \text{--- 5.82}$$

This quantity is independent of radius of sphere. Therefore the conservation of energy is expected.

5.7 Transmission Lines Theory

In an electronic system, the delivery of power requires the connection of two wires between the source and the load. At low frequencies, power is considered to be delivered to the load through the wire.

In the microwave frequency region, power is considered to be in electric and magnetic fields which are guided by some physical structure. Any physical structure that will guide an electromagnetic wave one place to other place is called a *Transmission Line*.

A transmission line is a device designedto guide electromagnetic energy from one point to another. For example, it is used to transfer the output rf energyof a transmitter to an antenna. This energy will nottravel through normal electrical wire without greatlosses. Although the antenna can be connecteddirectly to the transmitter, the antenna is usuallylocated some distance away from the transmitter. A transmission line is used to connect the transmitter and the antenna.The use of the transmission line purpose is totransfer the energy from the transmitter to theantenna with the least possible power loss. The transmission quality depends on the special physical andelectrical characteristics (impedance and resistance)of the transmission line.

5.7.1 Types of Transmission Lines

The types of transmission lines include

Coaxial cable

Two wire line

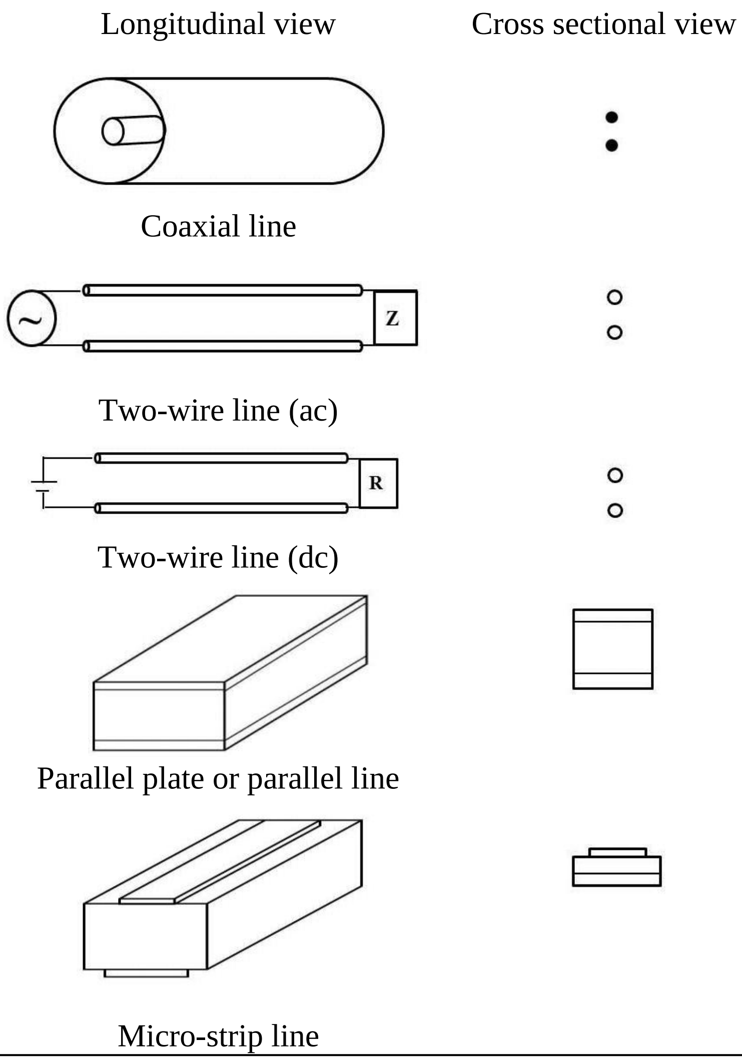
Parallel plate or parallel line

Micro-strip line

A wire above the conducting plane

Optical fiber

The different types of transmission lines are portrayed in Figure 5.10.



Wire above conducting plane

Optical fiber

Figure 5.6

The electric field is associated with the potential difference between the two ends of the conductor. When an electromagnetic wave propagates through the line, both the electric and magnetic fields are at right angles to each other. The plane containing both the fields is at right angle to the axis of the conductor. It means that the electric energy flows at right angle to the plane containing the electric and magnetic vectors. This type of transmission is technically known as transverse electric (TE) and transverse magnetic (TM) mode. This type of transfer of signal from transmitting end to the receiving end is called wire line connection. Thus the transmission line can be regarded as a guiding field so that it is confined near the line rather than spreading in space.

5.7.2 Electric current parameters

Transmission line constants, called distributedconstants, are spread along the entire length of thetransmission line and cannot be distinguished separately.The amount of inductance, capacitance, andresistance depends on the length of the line, the sizeof the conducting wires, the spacing between thewires, and the dielectric (air or insulating medium)between the wires.

Inductance of a transmission line

When current flows through a wire, magnetic linesof force are set up around the wire. As the currentincreases and decreases in amplitude, the field aroundthe wire expands and collapses accordingly. Theenergy produced by the magnetic lines of forcecollapsing back into the wire tends to keep the currentflowing in the same direction. This represents a certainamount of inductance (L), which is expressed inmicrohenrys per unit length.

Capacitance of a transmission line

Capacitance (C) also exists between the transmissionline wires. Notice thatthe two parallel wires act as plates of a capacitor andthat the air between them acts as a dielectric.

Thecapacitance between the wires is usually expressedin pico-farads per unit length. This electric fieldbetween the wires is similar to the field that existsbetween the two plates of a capacitor.

Resistance of a transmission line

The transmission line has electrical resistance (R) along its length. This resistanceis usually expressed in ohms per unit length and existing continuously from one end of theline to the other.

Leakage current

Since any dielectric, even air, is not a perfectinsulator, a small current known as leakagecurrent flows between the two wires. In effect, the insulator acts as a resistor, permitting current topass between the two wires. This property is called conductance (G) and is the opposite of resistance.Conductance in transmission lines is expressed as thereciprocal of resistance and is usually given inmicromhos per unit length.

5.8 Transmission line as distribution circuit

The equivalent circuit of a transmission line is shown in Figure 5.11. The inductance (L), resistance (R), conductance (G) and capacitance (C) are distributed uniformly along the entire length l of the transmission line. Each conductor of the line has certain length and diameter. So, it is said to be a uniform transmission line. The analysis based on this model is known as *"Distributed parameter"* model of a transmission line.

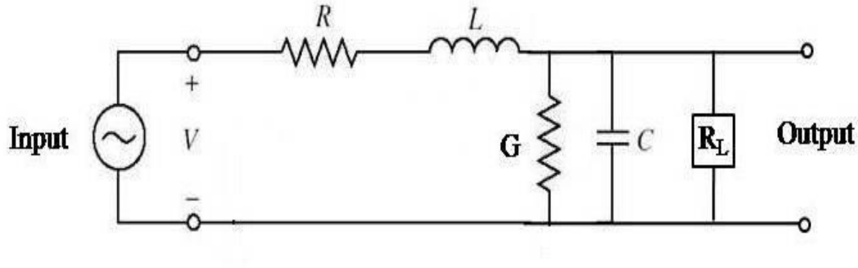


Figure 5.11

At radiofrequencies: The frequency of radio waves is very high, so the inductive reactance (ωL) is much larger than the resistance R i.e. ωL >> R. Also the capacitive substance (ωC) is larger to the shunt conductance 'G' i.e. ωC >> G. Here we can write the simplified equivalent circuit representation of a transmission line.

The are two ideal cases for any transmission line are

(a) infinite length ($l = \infty$) and

(b) lossless transmission line (i.e. R = 0 and G = 0)

5.9 Basic transmission line equations

The basic transmission line equations can be analysed either by using Maxwell's equations or distributed circuit parameter theory. The later has the more advantage than the former because it involves only one space variable and use one dimensional differential equations.

The current and voltage varies from point to point along the transmission line. The various notations used in this derivation are given below.

R = Series resistance of the line

L = Series inductance of the line

C = Series capacitance between the conductors

G = Shunt leakage conductance

ωL= Series reactance

ωC= Shunt capacitance

Z = R+jωL, Series impedance

Y = R+jωL, Shunt admittance

S = Distance measured form receiving end

I = Current in the line at any point

E = Voltage between any two points in the conductor

L = Length of the wire

The transmission line of length l can be made up of infinite T-sections as shown in Figure 5.12.

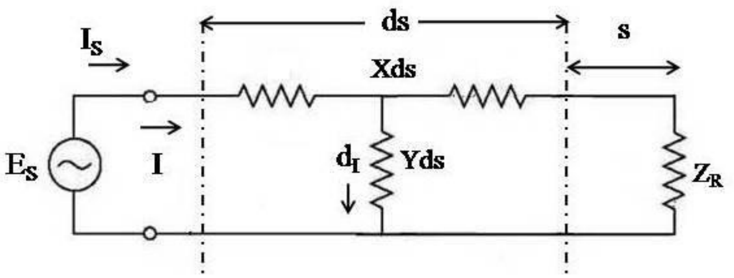


Figure 5.12

The point under consideration is at a distance S from the receiving end. The length of section is ds, hence its series impedance is Zds and shunt admittance is Yds. Here we assume that L is variable If C is variable then R, L and G are constants.

The condition for distortionless transmission can be derived as follows.

A line in which there is no phase or frequency distortion and it is correctly terminated is called distortionless line.

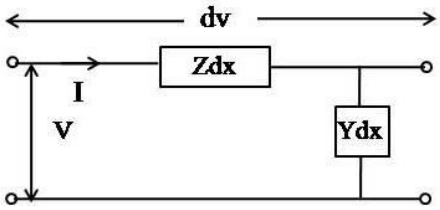


Figure 5.13

Transmission line with series impedance Z and shunt admittance Y is shown in Figure 5.13. Referring to Figure 5.13, we have in the more general case for sinusoidal variation of V and I with R and C are not zero. Hence a series impedance

$$Z = R + j\omega L \qquad \text{--- 5.83}$$

and a shunt admittance

$$Y = G + j\omega C \qquad \text{--- 5.84}$$

Where ω = 2πf, angular frequency

The square root of ZY is known as the propagation constant, γ, which may have real and imaginary parts. Hence the propagation constant (γ) is given by

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad \text{--- 5.85}$$

Squaring on both sides

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$

$$\gamma^2 = RG + j\omega RC + j\omega LG - \omega^2 LC$$

In order to get minimum attenuation, RC = LG, hence we have

$$\gamma^2 = RG - \omega^2 LC + j\omega RC + j\omega RC$$

$$\gamma^2 = RG - \omega^2 LC + 2j\omega RC$$

$$\gamma^2 = \sqrt{(RG)^2} - \left(\omega\sqrt{LC}\right)^2 + 2j\omega RC$$

$$\gamma^2 = \sqrt{(RG)^2} - \left(\omega\sqrt{LC}\right)^2 + 2j\omega\sqrt{RC\ LG} \quad \begin{bmatrix} RC = LG \text{ or } \sqrt{RC}\sqrt{RC} = \sqrt{LG}\sqrt{LG} \\ RC = \sqrt{RC}\sqrt{LG} = \sqrt{RC\ LG} \end{bmatrix}$$

$$\gamma^2 = \left[\sqrt{(RG)} - \left(\omega\sqrt{LC}\right)\right]^2 + 2j\omega\sqrt{RC\ LG}$$

$$\gamma^2 = \left[\sqrt{(RG)} - \left(j\omega\sqrt{LC}\right)\right]^2 \quad \left[\because (a-b)^2 = a^2 + b^2 - 2ab\right]$$

or

$$\gamma = \sqrt{(RG)} - j\omega\sqrt{LC}$$

or $\gamma = \alpha - j\beta$ --- 5.86

where α = real part i.e. attenuation constant = $\sqrt{(RG)}$

and β = imaginary part i.e. phase constant = $\omega\sqrt{LC}$

This is the required condition for attenuation less i.e. loss less transmission line.

5.3.1 Comparison of Waveguide and Transmission Line Characteristics

Transmission line	Waveguide
• Two or more conductorsseparated by some insulatingmedium (two-wire, coaxial, microstrip, etc.).	• Metal waveguides are typicallyone enclosed conductor filledwith an insulating mediumwhile a dielectric waveguide consists ofmultiple dielectrics.
• Normal operating mode is theTEM or quasi-TEM mode (cansupport TE and TM modes butthese modes are typically undesirable).	• Operating modes are TE or TMmodes (cannot support a TEMmode).
• No cut-off frequency for the TEMmode. Transmission lines cantransmit signals from DC up tohigh frequency.	• Must operate the waveguide at afrequency above the respectiveTE or TM mode cut-off frequencyfor that mode topropagate.
•Significant signal attenuation athigh frequencies due toconductor and dielectric losses.	• Lower signal attenuation at highfrequencies than transmission lines.
• Small cross-section transmissionlines (like coaxial cables) canonly transmit low power levelsdue to the relatively high fieldsconcentrated at specific locationswithin the device (field levels arelimited by dielectric breakdown).	• Metal waveguides can transmithigh power levels. The fields ofthe propagating wave are spreadmore uniformly over a largercross-sectional area than thesmall cross-section transmissionline.
• Large cross-section transmissionlines (like power transmissionlines) can transmit high powerlevels.	• Large cross-section (lowfrequency) waveguides areimpractical due to large size andhigh cost.

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